A characterization of Riemann integrability (HW 12)

The main work of this homework is to prove the following theorem.

THEOREM Suppose $f : [a, b] \to \mathbb{R}$ is a bounded function. Let $D = \{x \in [a, b] : f$ is discontinuous at x$\}$. Then the following are equivalent.

1. f is Riemann integrable.
2. D has measure zero.

The proof will be organized as Problmes 1-4 below. First are some definitions, and then some facts you can use.

Some definitions

DEFN Given x in $[a, b]$ define the oscillation of f at x to be

$$\begin{aligned} \operatorname{osc}(f)(x) &= \inf_{\delta>0} \sup\{|f(s) - f(t)| : |x - t| \leq \delta, |x - s| \leq \delta\} \\ &= \lim_{\delta\to 0^+} \sup\{|f(s) - f(t)| : |x - t| \leq \delta, |x - s| \leq \delta\} \end{aligned}$$

Given $\epsilon > 0$, define $D_\epsilon = \{x \in [a, b] : \operatorname{osc}(f)(x) \geq \epsilon\}$.

NOTE: $D = \cup_{\epsilon>0} D_\epsilon$.

DEFN A subset E of $\mathbb{R}$ has measure zero if for every $\epsilon > 0$ there is a countable collection of open intervals $\{(a_1, b_1), (a_2, b_2), \dots\}$, whose union contains E, such that $\sum_{n=1}^{\infty} b_n - a_n < \epsilon$.

NOT HARD TO CHECK: In the definition of measure zero, we can used closed intervals rather than open intervals, and we can also require the intervals to be pairwise disjoint.

Some facts

1. If X is a compact set in $\mathbb{R}$ and $\mathcal{C}$ is a collection of open sets which cover X (i.e. the union of the sets in the collection contains X), then there is a finite collection from $\mathcal{C}$ whose union covers X.
2. A union of countably many measure zero sets has measure zero. (More precisely: suppose $A_1, A_2, \dots$ are sets and each A_i has measure zero; then their union $\cup_i A_i$ has measure zero.)
3. f is integrable iff for all $\epsilon > 0$ there is a finite partition P such that $U(f, P) - L(f, P) < \epsilon$.

HW 12

Here is an outline of four steps to follow to prove the theorem.

Problem 1. Prove that if $\epsilon > 0$, then D_ϵ is a closed set.

Problem 2. Suppose $[c, d]$ is a closed interval and for every x in $[c, d]$, $\text{osc}(f)(x) < \epsilon$. Prove there is a partition $P = \{c = x_0 < x_1 < ... < n = d\}$ of $[c, d]$ such that $U(f, P) - L(f, P) < \epsilon(d - c)$.

Problem 3. Show (1) $\implies$ (2).
(Hint.
(i) Use one of the facts to show it is enough to prove for each positive integer n that $D_{1/n}$ has measure zero.
(ii) Given $\epsilon > 0$, let $P = \{a = x_0 < x_1 \cdots < x_n = b\}$ be a partition with $U(f, P) - L(f, P) < \epsilon$. Let $\mathcal{C}$ be the collection of subintervals $[x_{i-1}, x_i]$ with $M_i - m_i \geq 1/n$. Show $D_{1/n}$ is contained in the union of these subintervals. Get a bound on the sum of their lengths.)

Problem 4. Prove (2) $\implies$ (1).
(Hint: Suppose $\epsilon > 0$.
(i) Show D_ϵ has measure zero.
(ii) Show D_ϵ can be covered by a union of FINITELY MANY disjoint intervals $[a_1, b_1], \ldots, [a_k, b_k]$ whose lengths sum to less than ϵ. (List them in order, so $b_1 < a_2$, etc.)
(iii) Show that $\{a_1 b_1, a_2, b_2, \ldots, a_k, b_k\}$ can be enlarged to a finite partition P of $[a, b]$ such that $U(f, P) - L(f, P)$ is a function of ϵ which goes to zero as ϵ goes to zero.
(Hint: Let $[c, d]$ be any of the intervals $[a, a_1], [b_1, a_2], \ldots$. Apply part (ii) to $[c, d]$, using $\epsilon/(k + 1)$ in place of ϵ.)
(iv) Note that this completes the proof.

Problem 5. Suppose $f \colon [a, b] \to \mathbb{R}$ and f is integrable. Use the Theorem to prove that $|f|$ is integrable.
(Hint: compare the discontinuity sets of f and $|f|$.)