Proofs by Induction

Principle of mathematical induction. Suppose a set S satisfies the following two properties:

(1) The number 1 is in S.
(2) If x is in S, then $x+1$ is in S.

Then the entire set $\mathbb{N} = \{1, 2, 3, ...\}$ of postive integers is in S.

Sometimes induction is used to prove some proposition holds for ever positive integer. Let $P(1), P(2), P(3), ...$ denote a sequence of propositions, where $P(n)$ is a proposition about the positive integer n. The proof technique of mathematical induction proves $P(n)$ is true for each positive integer n by doing the following.

(1) (Basis Step) Show $P(1)$ is true.
(2) (Inductive Step) Assuming $P(n)$ is true (i.e. assuming the "inductive hypothesis"), show $P(n+1)$ is true.

Your homework problems using induction should make these steps clear. Here is an example.

Proposition 0.1. *Prove that the following equality $P(n)$ holds, for every positive integer n:*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \ .$$

Proof. We prove by induction.

Basis Step. $P(1)$ is true because $1 = (1)(1+1)/2$.

Inductive Step. Suppose $P(n)$; we will prove $P(n+1)$. The left hand side of $P(n+1)$ is

$$\begin{aligned}
\sum_{k=1}^{n+1} k &= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \quad \text{by the inductive hypothesis} \\
&= \frac{1}{2}n^2 + \frac{1}{2}n + n + 1 \\
&= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \ .
\end{aligned}$$

The right hand side of $P(n+1)$ is

$$\begin{aligned}
\frac{(n+1)(n+2)}{2} &= \frac{1}{2}(n^2 + 3n + 2) \\
&= \frac{1}{2}n^2 + \frac{3}{2}n + 1 \ .
\end{aligned}$$

Because the left and right sides are equal, $P(n+1)$ is true. This finishes the proof. $\square$

A remark: sometimes, the easiest way to show that two polynomials are equal is just to expand each one of them to the standard form (as we did above).

Another remark: there are many variations on the proof method of mathematical induction. Also there are various acceptable ways to write it down. However, you should be careful to address each step explicitly.