Math 410 – Spring 2016 – Boyle –Exam 1

There are 100 points possible.

1(a). (5 points) State the Extreme Value Theorem.
1(b). (5 points) State the Mean Value Theorem.
1(c). (6 points) State the positivity axioms for the real numbers.

2(a). (5 points) Suppose D is a subset of $\mathbb{R}$, $f\colon D \to \mathbb{R}$ and $x_0 \in D$. State the epsilon - delta condition for f to be continuous at x_0.
2(b). (5 points) Now define $D = [0, 2]$ with $f(x) = 5$ if $x \le 1$ and $f(x) = 7$ if $x > 1$. Prove that f does not satisfy the epsilon-delta condition for continuity at the point $x_0 = 1$.

3(a). (5 points) Compute $\liminf\ (-1)^n(2 - e^{-n})^3$. No proof required.
3(b). (5 points) Give an example of bounded sequences of real numbers, (a_n) and (b_n), for which $\limsup(a_n + b_n) \neq \limsup(a_n) + \limsup(b_n)$. State what the numbers $\limsup(a_n)$ and $\limsup(b_n)$ are in your example. No proof required.

4. (7 points) Let (C) denote the following condition on a sequence (b_n) of real numbers and a real number b: there exists $\epsilon > 0$ such that there exists N such that for all indices $n > N$, $|b_n - b| < \epsilon$.

Which of the rules (i),(ii),(iii),(iv) below define sequences $(b_n)_{n=1}^{\infty}$ which satisfy condition (C) for some real number b? No proof required.
(i) $b_n = 4 + (8/n)$
(ii) $b_n = (-1)^n(4 + (8/n))$
(iii) $b_n = 1$ if $n \le 23$ and $b_n = 2$ if $n > 23$
(iv) $b_n = \sqrt{n}$

5. (14 points) Let k and c be real numbers. Suppose $F : \mathbb{R} \to \mathbb{R}$ and F satisfies the conditions

$$F'(x) = kF(x)\ , \qquad F(0) = c\ .$$

Prove that $F(x) = ce^{kx}$, for all x.

****There are more problems on the other side. ****

6. (15 points) Suppose $-\infty < a < b < \infty$ and $f\colon [a,b] \to \mathbb{R}$ is continuous. Prove that f is uniformly continuous.

7. (28 points) For each of the following, answer TRUE or FALSE. In the case that the correct answer is FALSE, give a counterexample (without proof, please). There are 4 points for each part.

(a) If E is a nonempty bounded set of real numbers, then $\sup E$ is a limit point of E.

(b) If D is an open interval and $f\colon D \to \mathbb{R}$ is a strictly increasing differentiable function onto its image $f(D)$, then $f^{-1}\colon f(D) \to D$ is a strictly increasing differentiable function.

(c) If a, b, c and d are nonzero real numbers, then the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a + bx + cx^2 + dx^9$ has a real root.

(d) If f and g are continuous functions from $\mathbb{R}$ to $\mathbb{R}$ and $f(x) = g(x)$ for every irrational number x, then $f(x) = g(x)$ for every x.

(e) If $f : [7,9] \to \mathbb{R}$, $f(8) = 6$ and

$$\lim_{x\to 8} \frac{\Big(f(x) - (2x - 10)\Big)}{x - 8} = 0 \quad ,$$

then f is differentiable at $x = 8$.

(f) If $f : \mathbb{R} \to \mathbb{R}$, $f(0) = 1$ and $f(x) = x$ for nonzero x, then $\lim_{x\to 0} f(x)$ does not exist.

(g) A bounded continuous function from $(0,1)$ into the real numbers is uniformly continuous.