Math 410 – Fall 2012 – Boyle –Exam 2

1. (a) (8 points) Suppose $f : [0,1] \to \mathbb{R}$ and $f(x) = 0$ except at finitely many inputs x. Prove that f is integrable and $\int_{x=0}^{1} f(x)\,dx = 0$.
   (b) (2 points) Suppose f and g are functions from $[0,1]$ to $\mathbb{R}$; f is integrable; and $f(x) = g(x)$ except at finitely many inputs x. Prove that g is integrable and $\int_0^1 g = \int_0^1 f$.

2. (12 points) Suppose f is a continuous function and $F(x) = \int_{t=0}^{x} f(t)\,dt$. Assuming $x > 0$, prove

$$\lim_{h \to 0^+} \frac{F(x+h) - F(x)}{h} = f(x) \ .$$

   (Don't prove this by citing the Fundamental Theorem of Calculus – you are asked here to show part of the proof.)

3. (8 points) Compute the following limits. Show a little work.
   (a) $\lim_{x \to 0^+} x(\ln x)$
   (b) $\lim_{x \to 0} x^{-10} e^{-1/|x|}$

4. Suppose $f : [a,b] \to \mathbb{R}$ is a bounded function.
   (a) (6 points) For a partition $P = \{x_0 < x_1 < \cdots < x_n\}$ of $[a,b]$, give the definition for the lower and upper sums $L(f,P)$, $U(f,P)$.
   (b) (2 points) Give the definition for the lower and upper integrals $\underline{\int_a^b} f$ and $\overline{\int_a^b} f$.
   (c) (4 points) Give an example f for which the lower and upper integrals differ.

5. (8 points) For the function $g(x)$ below, compute $g'(3)$.

$$g(x) \ = \ \int_{t=e}^{x} \cos(\ln(t))\,dt \quad + \quad \int_{t=x^2}^{e^\pi} \cos(\ln(t))\,dt \ .$$

6. (a) (3 points) Give the definition of the nth Taylor polynomial p_n of a sufficiently differentiable function f at a particular input x_0.
   (b) (3 points) State the Lagrange Remainder Theorem.
   (c) (2 points) What is the Taylor series (with respect to $x_0 = 0$) of the function $f(x) = \sin x$?
   (d) (8 points) Find a polynomial p such that $|\sin(x) - p(x)| \le 1/100$ for all x in $[0,2]$. Prove the approximation holds.

7. (10 points) For the following functions, compute the Taylor polynomial $p_4(x)$ at $x_0 = 0$.
(a) $f(x) = (\cos(3x))(\sin x)$.
(b) $g(x) = (1 + x^3)e^x$.

8. (24 points) Answer each question below TRUE or FALSE. No justification required.

   (a) If $f : [a, b] \to \mathbb{R}$ is monotone, then f is integrable.

   (b) If $f : [a, b] \to \mathbb{R}$ is integrable and for $a < x < b$ we define $g(x) = \int_{t=a}^{x} f(t)\,dt$, then $g'(x) = f(x)$.

   (c) If $f : [a, b] \to \mathbb{R}$ is Lipschitz, then f is integrable.

   (d) For bounded functions f and g from $[a, b]$ to $\mathbb{R}$,

$$\overline{\int_a^b} f + \overline{\int_a^b} g \;\leq\; \overline{\int_a^b} f + g \ .$$

   (e) Suppose $f : \mathbb{R} \to \mathbb{R}$ has derivatives of all orders at every point, and p_n is the nth Taylor polynomial of f at the point $x_0 = 0$. Then $\lim_{n\to\infty} p_n(x) = f(x)$, for all x in $\mathbb{R}$.

   (f) If n is a positive integer and the nth derivative of f exists at x_0 and p_n is the nth Taylor polynomial of f at x_0, then

$$\lim_{x\to x_0} \frac{f(x) - p_n(x)}{(x - x_0)^n} \;=\; 0 \ .$$

   (g) A bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable with integral r if and only if the following property holds: for every Riemann sum $R(f, \mathcal{P}, C)$, for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\text{gap}(\mathcal{P}) < \delta \implies |R(f, \mathcal{P}, C) - r| < \epsilon \ .$$

   (h) If a function $f : [a, b] \to \mathbb{R}$ is bounded, then for every partition $\mathcal{P}$ of $[a, b]$ and every associated Riemann sum $R(f, \mathcal{P}, C)$,

$$L(f, P) \leq R(f, \mathcal{P}, C) \leq U(f, P) \ .$$