Uniform continuity: a worked problem

Problem 4 in Section 9.3 We are given $f_n: [0,1] \to \mathbb{R}$, defined by $f_n(x) = x/(nx+1)$. We're asked to find the limit function f, and prove that the convergence is uniform.

It is easy to see that f_n converges pointwise to the function f(x) = 0 for all x. (The proof below of uniform convergence also proves this pointwise convergence.)

To show the convergence is uniform, suppose $\epsilon > 0$. We must find N such that for all x in [0, 1],

$$n \ge N \implies |f_n(x) - 0| < \epsilon$$
.

So, let N be an integer greater than $1/\epsilon$, and suppose $n \ge N$. If $\epsilon \le x \le 1$, then

$$|f_n(x)| = \frac{x}{nx+1} < \frac{x}{nx} = \frac{1}{n} \le \frac{1}{N} < \frac{1}{(1/\epsilon)} = \epsilon$$

On the other hand, if $0 \leq x < \epsilon$, then

$$|f_n(x)| = \frac{x}{nx+1} < \frac{x}{1} = x < \epsilon$$
.

This completes the proof.

Note that for the proof, we needed different arguments (depending on x) to justify the inequality. Estimates are fundamental to analysis (the branch of mathematics which includes calculus), and it's very common that more than one kind of argument is needed in a problem to establish an estimate. A theme of analysis:

"Divide and conquer."