Two views of continuity

Proposition F. Suppose f is a function from a subset D of $\mathbb{R}$ into $\mathbb{R}$, and x is in the domain D of f. Then the following are equivalent.

1. If (x_n) is a sequence from D and $\lim_n x_n = x$, then $\lim f(x_n) = f(x)$.
2. For every $\epsilon > 0$, there exists $\delta > 0$ such that for all w in D,

$$|w - x| < \delta \implies |f(w) - f(x)| < \epsilon \ .$$

Definition. If f and x satisfy one (hence both) of the conditions above, then we say that f is *continuous at the point* x.

Definition. If f is continuous at every point in its domain, then we say that f is continuous.

For context we remark that there is another way (the most general way) to characterize a continuous function, using open sets, but we will not go into it here.

Now we prove Proposition F.

Proof for (2) $\implies$ (1).
Suppose (x_n) is a sequence from D and $\lim_n x_n = x$. We must prove that $\lim f(x_n) = f(x)$. For this, suppose $\epsilon > 0$. We must show there is an N such that $n \geq N$ implies $|f(x_n) - f(x)| < \epsilon$.
Since (2) holds, we have a $\delta > 0$ such that $|x_n - x| < \delta$ implies $|f(x_n) - f(x)| < \epsilon$. Because $\lim_n x_n = x$, we have N such that $n \geq N \implies |x_n - x| < \delta$. Combining the facts, we have as required that

$$n \geq N \implies |f(x_n) - f(x)| < \epsilon \ .$$

Proof for (1) $\implies$ (2).
We will prove an equivalent statement: if (2) is false, then (1) is false. So suppose (2) is false. Then there is an $\epsilon > 0$ such that there is no positive δ which guarantees that

$$|w - x| < \delta \implies |f(w) - f(x)| < \epsilon \ .$$

For this ϵ, and for $n \in \mathbb{N}$, we can then pick x_n from D such that $|x_n - x| < 1/n$ and also $|f(x_n) - f(x)| \geq \epsilon$. But then, $\lim_n x_n = x$ and $\lim_n f(x_n) \neq f(x)$. Therefore (1) is false. **QED**