THE WORK OF KIM AND ROUSH IN SYMBOLIC DYNAMICS

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Contents

1. Introduction

About one fifth of the papers of Hang Kim were joint papers with Fred Roush addressing problems of symbolic dynamics. They were remarkable problem solvers who made huge contributions to the topic. This paper reviews those contributions, at the level of describing main results and giving some context. There isn't room for the best part – the ideas behind the results – but I hope this survey is of some use for appreciating the contributions and knowing where to look to find more.

The bibliography of references to their work lists papers in chronological order; the supplementary bibliography is ordered alphabetically by author.

Many background definitions are omitted. Most basic background can be easily found in the very accessible book of Lind and Marcus [55]. I use the notation σ_A to represent a shift of finite type defined by a square nonnegative integral matrix A. For conciseness, I don't try to reconcile notation or terminology with statements in the various papers. For exact statements one can consult the originals.

2. Decidability results

Shift equivalence. The first work of Kim and Roush for symbolic dynamics, begun in the 1979 paper [1] and completed in [3], was to prove that there is a decision procedure which takes two square matrices over $\mathbb{Z}_+$ and determines whether they

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are shift equivalent over $\mathbb{Z}_+$. Williams [60] show that if A and B are shift equivalent over $\mathbb{Z}_+$, then σ_A and σ_B are *eventually conjugate* (i.e., σ_A^n and σ_B^n are topologically conjugate for all but finitely many n). Kim and Roush proved the converse holds [1]. Shift equivalence is a very strong equivalence relation, and a very important one in the study of shifts of finite type.

Stationary dimension groups and modules. This subsection gives some definitions to set notation used later.

Suppose A is an $n \times n$ matrix over $\mathbb{Z}_+$. We need to fix a convention of having A act on row vectors or column vectors; we choose rows. Then let G_A denote the direct limit group defined from A by the action on $\mathbb{Z}^n$,

$$G_A \;=\; \varinjlim_{A} \mathbb{Z}^n \;=\; \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \mathbb{Z}^n \xrightarrow{A} \cdots \quad .$$

An element of the direct limit group is an equivalence class of elements $\{(v,k) : k \in \mathbb{Z}_+, v \in \mathbb{Z}^n\}$, with equivalence classes generated by the relation $(v,k) \sim (vA, k+1)$ for all v and k. So, $[(v,k)] = [(w,j)]$ iff $vA^{n+j} = wA^{n+k}$ for some (hence all large) $n \geq 0$. The group G_A becomes an ordered group (specifically, a stationary dimension group) by the definition that the positive set G_A^+ is the set of $[(v,k)]$ such that $vA^m \geq 0$ for some $m \geq 0$ (equivalently, the set $[(v,k)]$ contains some (w,j) such that $w \geq 0$). The matrix A induces an automorphism $\widehat{A}$ of this ordered group, defined by $\widehat{A} : [(v,k)] \mapsto [(vA,k)]$. The ordered group then becomes an ordered module, $\mathcal{M}_A$, over the Laurent polynomials $\mathbb{Z}[t,t^{-1}]$, with t acting by $t : [(v,k)] \mapsto [(v,k+1)]$, and t^{-1} acting by $\widehat{A}$. We call $\mathcal{M}_A$ the dimension module of A.

For an SFT σ_A, Krieger used a Groethendieck style topological construction to associate $\mathcal{M}_A$ canonically to σ_A. He showed that matrices A, B are shift equivalent over $\mathbb{Z}_+$ if and only if they have isomorphic dimension modules.

The most important class of SFT are the mixing SFTs. These are the SFTs which are topologically conjugate to some σ_A for which A is a primitive matrix A. For A, B primitive, $\mathcal{M}_A$ and $\mathcal{M}_B$ are isomorphic as ordered modules if and only if they are isomorphic as unordered modules, and therefore in this case the order structure is often disregarded.

When A is primitive, the ordered group (G_A, G_A^+) is a *stationary simple dimension group.*

Closing maps and module epimorphisms. One of the basic coding questions around shifts of finite type is the following: given mixing SFTs σ_A, σ_B is there a factor map from σ_A onto σ_B? The question is answered in the case the SFTs have different entropy. When both have equal entropy, the question is open, but there is a very strong partial result due to Ashley [37]: there is a "right closing" factor map from σ_A to σ_B if and only there is an epimorphism from the dimension module of A to that of B. The factor maps which are right closing or left closing are a large and useful class, and they are the only factor maps between SFTs of equal entropy which we know how to construct in some generality [35, 37, 38, 46, 55]. The distinction between right and left closing maps can be described at the matrix level as the distinction between considering the action of a matrix A on row vectors or column vectors.

Given primitive A and B with integer entries, Kim and Roush [4] gave an algorithm which produces an epimorphism from the dimension module for A to that of B, or determines that none exists.

Stationary simple dimension groups and C^*-algebras. Bratteli, Jorgensen, Kim and Roush [29, 32] gave a decision procedure for determining whether two stationary simple dimension groups are isomorphic. They also gave a variety of techniques for computation in special cases [31]. The motivation for this work came from C^*-algebras: two AF C^*-algebras are stably isomorphic if and only if their associated dimension groups as ordered groups are isomorphic. Thus the decidability result showed that stable isomorphism of AF C^*-algebras with stationary Bratteli diagrams is decidable. The decidability result is also meaningful directly to symbolic dynamics. For example, a necessary condition for topological orbit equivalence of two mixing shifts of finite type σ_A, σ_B is that the simple dimension group generated from each by the range of the measure of maximal entropy on cylinder sets must be the same [42].

The dimension groups of A and B can be isomorphic even if their dimension modules are not [24]. The order structure of a simple stationary dimension module carries no information as an invariant, given the module structure. But, given the group structure without the module structure, the order structure is a very rich additional invariant.

Other decidability results of Kim and Roush are described in the Boolean and sofic sections below. Kim and Roush also proved significant decidability results outside of symbolic dynamics, which are reviewed in the article of Kirsten Eisenträger [48].

3. Shift and strong shift equivalence for Boolean matrices

Let $\mathcal{B}$ be the Boolean semiring on two elements, which we denote $\{0, 1\}$. Here 0 is the additive identity; 1 is the multiplicative identity; and $1 + 1 = 1$. Because $\mathcal{B}$ is the quotient of the semiring $\mathbb{Z}_+$ under the map sending positive integers to 1 and zero to 0, the structure of shift equivalence and strong shift equivalence of matrices over $\mathcal{B}$ is relevant to constructions and decision procedures for nonnegative matrices over Z and other subrings of $\mathbb{R}$. A significant part of their initial work [1] on decidability of shift equivalence of matrices over $\mathbb{Z}_+$ involved the study of their structure as Boolean matrices.

Kim and Roush showed that all primitive matrices are shift equivalent over $\mathcal{B}$, but primitive matrices A, B over $\mathcal{B}$ are strong shift equivalent over $\mathcal{B}$ if and only if $\text{tr}(A^n) = \text{tr}(B^n)$ for all positive integers n [2]. (In particular, for primitive Boolean matrices, shift equivalence does not imply strong shift equivalence.) They showed also [2] that for a matrix A over a dense subring $\mathbb{S}$ of $\mathbb{R}$, any SSE of the Boolean image of A to a Boolean matrix B can be mimicked by an SSE over $\mathbb{S}_+$ of A to matrix with a block pattern of positive and zero blocks matching the matrix B. In particular, every primitive matrix over $\mathbb{S}$ with positive trace is SSE over $\mathbb{S}_+$ to a positive matrix (meaning, an entrywise positive matrix). This "positive matrix lemma" was useful for later work, as indicated below.

4. Strong shift equivalence of positive matrices over subrings of $\mathbb{R}$

To this day, we do not know a general sufficient condition for strong shift equivalence of matrices over $\mathbb{Z}_+$. For example, it is not known whether there exist algorithms for the following problems:

- Given 2×2 matrices over $\mathbb{Z}_+$, determine whether they are SSE over $\mathbb{Z}_+$.
- Given primitive matrices A, B over $\mathbb{Z}_+$, each with just one nonzero eigenvalue, which is the same, determine whether they are SSE over $\mathbb{Z}_+$.

In the 1990-1992 series of papers [6, 10, 12, 14], also appealing to results from the Boolean paper [2], Kim and Roush developed path methods to give sufficient conditions for proving SSE of matrices over dense subrings of $\mathbb{R}$. The central result is the following: if A and B lie on a path of positive matrices from a single conjugacy (similarity) class, then A and B are strong shift equivalent over $\mathbb{R}_+$. For a dense subring $\mathbb{S}$ of $\mathbb{R}$, by their "positive matrix lemma" mentioned above they could immediately reduce the question of strong shift equivalence of primitive matrices over $\mathbb{S}$ to the question of strong shift equivalence over positive matrices over $\mathbb{S}$. For positive matrices, perturbation methods could be applied to deduce theorems over some rings $\mathbb{S}$ as corollaries of results over $\mathbb{R}$. In particular, Kim and Roush were then able to prove that two primitive matrices over a subfield $\mathbb{S}$ of $\mathbb{R}$, with the same spectral radius and with no other eigenvalue, are SSE over $\mathbb{S}$. This result remains in dramatic contrast to the lack of a general sufficient condition for strong shift equivalence over $\mathbb{Z}_+$.

A generalized version of essentially the entire theory developed in the cited papers is presented in [33] with complete supporting proofs and new results.

5. Automorphisms of the shift

The automorphism group $\mathrm{Aut}(\sigma_A)$ of an SFT σ_A is the countable group of homeomorphisms commuting with σ_A. The study of automorphisms of the shift was inaugurated by Hedlund [53], in the case of full shifts. As a probe to the classification of shifts of finite type, and as a test of his conjecture that shift equivalence implies strong shift equivalence for matrices over $\mathbb{Z}_+$ [60], Williams posed the following question: if a mixing SFT has two fixed points, does it have an automorphism which exchanges them? This evolved [45] into the Finite Extension Problem: determine when an automorphism of a finite subsystem of a mixing SFT is the restriction of an automorphism of the shift.

An automorphism U of an SFT σ_A induces a canonical automorphism $\widehat{U}$ of the dimension module of σ_A. (Here "induces a canonical" can be explained by Krieger's original construction or by an argument of Wagoner in the setting of his Strong Shift Equivalence Space [58].) The map $U \mapsto \widehat{U}$ defines a group homomorphism $\rho_A : \mathrm{Aut}(\sigma_A) \to \mathrm{Aut}(\mathcal{M}_A)$ which is called the *dimension representation* for the SFT σ_A. The *inert* automorphisms of σ_A are the automorphisms in the kernel of ρ_A.

The first (and by far more important) part of the Finite Extension Problem is to determine how the inert automorphisms can act on finite subsystems. The second part (still open in general), understanding how a noninert automorphism can act, to a large extent amounts to determining the range of the dimension representation [44, 45]. Roughly: the action of the kernel on finite subsystems is qualitatively much the same for different mixing SFTs; the kernel is the large, combinatorial and poorly understood part of the automorphism group. The action on the dimension group varies from SFT to SFT, involves algebra and is well understood in many cases [45].

Constraints on inert automorphisms. The automorphism group of an SFT has its dimension representation, and also a second representation which has been useful. This is the sign-gyration-compatibility homomorphism (SGCC), a map from $\mathrm{Aut}(\sigma_A)$ into $\prod_{n\in\mathbb{N}} \mathbb{Z}/n\mathbb{Z}$. For $U \in \mathrm{Aut}(\sigma_A)$, the nth coordinate of $\mathrm{SGCC}(\mathcal{U})$ is determined by the action of U on points in σ_A-orbits of length less than or equal to n.

Following earlier work by several authors, a central issue emerged: must SGCC vanish on inert automorphisms? Kim and Roush proved this in their 1991 paper [9], with an appeal to a "positive cocyles" lemma of Wagoner. This paper established the first constraints on the action of the automorphism group of a mixing shift of finite type on its finite subsystems, and in particular at last resolved the Williams fixed point question (in the negative). Along with earlier work, it also proved that not all inert automorphisms are compositions of inert automorphisms of finite order.

Sharper constraints on inert automorphisms. Kim, Roush and Wagoner [17] followed [9] with a much sharper result, providing explicit formulae for computing the SGCC homomorphism from the entries of matrices giving a strong shift equivalence. This argument takes place within the Strong Shift Equivalence Space algebraic topological framework developed by Wagoner.

The range of the dimension representation. A necessary ingredient for the complete classification of the possible actions of automorphisms of an SFT σ_A on finite subsystems is a characterization of the range of the dimension representation for σ_A. As part of their paper [17], Kim, Roush and Wagoner gave an example of a mixing SFT whose dimension module (as an ordered module) admitted an automorphism which could not be in the range of the dimension representation.

Constructed actions by inert automorphisms. The characterization of the action of the group of inert automorphisms on finite subsystems of a mixing SFT was completed when Kim, Roush and Wagoner proved that vanishing SGCC is the *only* obstruction to extending an automorphism of a finite subsystem to an automorphism of the shift [21,25,26]. Constructions had been made in many rather general cases by earlier authors; however, the Kim-Roush-Wagoner work handling the final remaining cases was extremely difficult, and introduced a new general techniqe for constructing topological conjugacies between SFTs. This technique is the heart of the "positive K-theory" discussed below.

Earlier counterexamples. In [11] Kim and Roush gave an example of a mixing SFT of entropy strictly less than $\log 2$ with an automorphism interchanging two fixed points. This example could not be simple in the sense of Nasu. It therefore refuted two conjectures of Wagoner [57] and clarified the structure of the Finite Extension Problem.

Subgroups of the automorphism group. Hedlund in [53] showed that the automorphism group of a full shift contained copies of all finite groups. This was generalized to mixing shifts of finite type and extended to some other classs of group in [45]. Among other results, in [7] Kim and Roush show that for every nontrivial mixing shift of finite type the following groups embed as subgroups of the automorphism group: any countable locally finite residually finite group; any graph group; the fundamental group of any 2-manifold; the automorphism group of any full shift. They also proved the following. Let a_n denote the number of automorphisms of a mixing shift of finite type with positive entropy $\log \lambda$ which have a defining block code depending only on coordinates $\{0, 1, \dots, n-1\}$. Then $\lim_n (1/n)(\log \log a_n) = \log \lambda$.

Automorphisms of onesided shifts and λ matrices. The paper [13] includes a study of inert automorphisms of onesided shifts of finite type and the quotient groups for a filtration of onesided inert automorphisms. There is a related, nearly contemporaneous paper by Ashley [36].

6. The nonzero spectra of nonnegative integral matrices

Since at least the 1940s, there has been work on the following problem: what can be the spectrum of a real matrix with nonnegative entries? A variant asks, what can be the nonzero spectrum of a nonnegative matrix? For precision, given an n-tuple $\Lambda = (\lambda_1, \dots, \lambda_n)$ of complex numbers, let $p_\Lambda(t)$ denote the polynomial $\prod_{i=1}^n (t - \lambda_i)$. Then Λ is the nonzero spectrum of a matrix A if its characteristic polynomial is $t^k p_\Lambda(t)$ for some $k \geq 0$.

The original problem reduces to knowing for primitive matrices the possible nonzero spectra and the minimal number of zeros which must be added for realization [8]. For $\mathbb{S}$ a unital subring of $\mathbb{R}$, the Spectral Conjecture of [8] conjectured that an n-tuple $\Lambda = (\lambda_1, \dots, \lambda_n)$ of complex numbers is the nonzero part of the spectrum of a primitive matrix with entries from $\mathbb{S}$ if Λ satisfies three simple necesssary conditions. Let $\mathrm{tr}(\Lambda^k) = \sum_{i=1}^n (\lambda_i)^k$, and let $\underline{\mathrm{tr}}_n = \sum_{k:k|n} \mu(n/k)\mathrm{tr}(\Lambda^k)$, where μ is the Mobius function. Then the three conditions are the following.

1. There is an i such that $\lambda_i > |\lambda_j|$ if $j \neq i$.
2. The polynomial p_Λ has its coefficients in $\mathbb{S}$.
3. (a) If $S \neq \mathbb{Z}$, then for all n and k in $\mathbb{N}$,
   - $\mathrm{tr}(\Lambda^n) \geq 0$, and
   - $\mathrm{tr}(\Lambda^n) > 0 \implies \mathrm{tr}(\Lambda^{nk}) > 0$.

   (b) If $S = \mathbb{Z}$, then for all n in $\mathbb{N}$, $\underline{\mathrm{tr}}_n \geq 0$.

The conjecture was proved in [8] for $\mathbb{S} = \mathbb{R}$, and in many other cases, but not in general, and especially not for $\mathbb{Z}$, the most difficult case.

Kim, Ormes and Roush proved the conjecture for the case $\mathbb{S} = \mathbb{Z}$ in [27].

This result has the consequence for symbolic dynamics of providing a characterization of the zeta functions of mixing SFTs, because the zeta functions of mixing SFTs are the reciprocals of the polynomials $\prod_{i=1}^n (1 - \lambda_i z)$ such that $(\lambda_1, \dots, \lambda_n)$ is the nonzero spectrum of a primitive integral matrix. By routine reductions, one then knows the possible zeta functions for all SFTs [8].

In an appendix by Boyle, Handelman, Kim and Roush to the paper [8], the verification of the Spectral Conjecture is extended from $\mathbb{R}$ to all $\mathbb{S}$ not equal to $\mathbb{Z}$ in the case that $\mathrm{tr}(\Lambda) > 0$ (i.e., the candidate spectrum has positive trace). A key ingredient of that argument is the "positive matrix lemma" mentioned in the Boolean section above.

7. The classification problem for shifts of finite type

Reducible SFTs. In 1974, Williams conjectured that for matrices over $\mathbb{Z}_+$, shift equivalence over $\mathbb{Z}_+$ implies strong shift equivalence over $\mathbb{Z}_+$. This conjecture was finally refuted in the 1992 paper [16] of Kim and Roush, which gave a counterexample consisting of two reducible matrices. Their argument involved a clever appeal to their proof with Wagoner [17] that the dimension representation for a mixing shift of finite type in general need not be surjective.

General SFTs via irreducible SFTs. Enlarging on the idea behind their reducible counterexample, Kim and Roush described how the classification problem for general (possibly reducible) SFTs could be reduced to two problems: the problem of classifying mixing SFTs up to topological conjugacy, and the problem of determining the range of the dimension representation for mixing SFTs [15].

Irreducible SFTs. The Williams Conjecture in the case of mixing SFTs was finally refuted, by Kim and Roush, in their 1999 Annals of Math paper [20,22]. This argument, like the automorphisms paper [17] with Wagoner, takes place in the framework of Wagoner's Strong Shift Equivalence Space. Wagoner gave another counterexample argument, using K_2 of the dual numbers, in [28], to which Kim and Roush contributed an appendix. The reviews [23] and [59] of the classification problem for shifts of finite type give a nice overview of all this, with open problems.

8. Classification of free $\mathbb{Z}_p$ actions on mixing SFTs

Let G be a finite group. A G-SFT is an SFT with a continuous G action which commutes with the shift. These are natural objects of study for symbolic dynamics, which have been approached in different ways [34, 47, 49]. Let $\mathbb{Z}_p$ denote the group $\mathbb{Z}/p\mathbb{Z}$. A natural question then is whether, given a prime p and a mixing SFT S, there is a free $\mathbb{Z}_p$ action on the SFT, commuting with S. This seems to be a question for another generation.

What Kim and Roush did do in [19] is to answer a closely related question. Let $G = \mathbb{Z}_p$. The orbit space of a free G-SFT Y is the SFT whose points are the G-orbits in Y. Given a mixing SFT X, they determined easily computed necessary and sufficient conditions (involving only the zeta function of X) for there to exist a G-SFT Y, shift equivalent as an SFT to X, such that the orbit space of Y is X. Further progress will probably require a deeper understanding of how shift equivalence refines strong shift equivalence over $\mathbb{Z}_+$.

A very interesting feature of [19] is that the proof requires the proof technique method of "positive K-theory" introduced in [25, 26]. In both cases, the technique makes possible a difficult proof, where without it we would have no proof at all.

9. Topological orbit equivalence of shifts of finite type

One of the dramatic developments in topological dynamics over the last two decades was the emergence of topological orbit equivalence and ordered cohomology in the study of Cantor minimal systems and their associated C^* crossed product algebras (e.g. [50, 51]). The topological orbit equivalence and ordered cohomology were then studied for shifts of finite type in [42] and the paper [30] of Kim, Roush and Susan Williams, who introduced several new tools, including a dual notion of ordered homology. They also showed that a homeomorphism between SFTs resepecting periodic points and their periods need not be a topological orbit equivalence (i.e., need not induce a bijection of orbits). They also showed for mixing SFTs that an isomorphism of ordered cochain groups respecting coboundaries is equivalent to existence of a homeomorphism between them which sends orbit closures to orbit closures. Still open is the question of whether the unital ordered cohomology is a complete invariant of orbit equivalence of mixing SFTs up to flip conjugacy.

Here is an unusual result from [30]. Suppose $n > 3$ and S is an irreducible SFT and S_n is the full shift on n symbols. Let R be the integral group ring of the automorphism group of the S_n. Then the unital ordered cohomology of the SFT $S \times S_n$, as an R-module, determines the topological conjugacy class of S.

It will be remarkable if this result becomes useful, since the the ordered cohomology of an SFT is fantastically complicated, and there is no classification known for either the automorphism group or the ordered cohomology. However the formulation of such an invariant reflects the energy and audacity of the authors (and perhaps a certain sense of humor).

10. Sofic shifts

Recall from Section 2 that Kim and Roush gave a decision procedure for determining whether two shifts of finite type are eventually conjugate. They did the same for sofic shifts.

We'll give a very sketchy background for this. The sofic shifts are the subshifts which are topological factors (quotients) of shifts of finite type. A fundamental step toward their classification up to topological conjugacy was taken by Krieger [54], who showed that certain SFT covers (derived from follower or predecessor sets) were canonically associated to a sofic system: a conjugacy between two sofic shifts lifts to a unique conjugacy of its right Krieger cover (constructed from follower sets), and the same holds for the left Krieger cover (constructed from predecessor sets).

Nasu [56] introduced a notion of strong shift equivalence of "representation matrices" for SFT covers, and with Hamachi [52] proved two covers have strong shift equivalent "representation matrices" if and only if the covers are topologically conjugate. So, for a class of canonical covers, topological conjugacy of the sofic shifts is equivalent to topological conjugacy of the representation matrices of the covers. These ideas were extended to a notion of shift equivalence for sofic shifts [43].

Let G be the semigroup (under matrix multiplication) of zero-one, $\mathbb{N}\times\mathbb{N}$, finitely supported, zero one matrices which have at most a one 1 in each row. Let $R_+ = \mathbb{Z}_+G$ denote the semiring inside the integral semigroup ring of G consisting of formal integral combinations with all coefficients nonnegative integers, modulo multiples of the zero matrix in G. Define shift equivalence and strong shift equivalence of elements in R_+ using the defining equations of shift equvalence and strong shift equjivalence. (We do not consider matrices with entries from R_+ – just elements of R_+.) To any sofic shift S, one can associate an element M_S of R_+, derived from the right Krieger cover. Now sofic shifts S, T are topologically conjugate if and only if M_S and M_T are strong shift equivalent in R_+; and the sofic shifts S, T are eventually conjugate if and only if M_s and M_T are shift equivalent in R_+.

Kim and Roush proved that shift equivalence in R_+ is decidable [5].

In [18], Kim and Roush introduced a dimension group for sofic shifts (which is a dimension group in the usual sense, together with additional structure) whose isomorphism class is a complete invariant for shift equivalence of the sofic shifts. Again, this developed the theory for sofic shifts to parallel that for shifts of finite type.

Examples of near Markov shifts were given in [18] to show that eventual conjugacy does not imply conjugacy and that such a Markov shift need not be equivalent to its time reversal.

11. Positive K-theory

Williams introduced the framework of strong shift equivalence of matrices over $\mathbb{Z}_+$ provide a framework for studying the classification of shifts of finite type and the isomorphisms (topological conjugacies) between them. In [25, 26], Kim, Roush and Wagoner introduced a new kind of basic unit for topological conjugacies between shifts of finite type σ_A and σ_B: for example, under some conditions, if E is a basic elementary matrix with off-diagonal entry a monomial in t, and $E(I-tA) = I-tB$, then there is induced a topological conjugacy from σ_A to σ_B. This is expanded to a classification theory emulating Williams' theory in [39] (also see [40]).

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