Review problems for the final exam

This is a selection of problems from exams given at the University of Maryland in the past few years. The exam may include material that is not on the review and the wording of questions may be somewhat different. Make sure that you do all suggested homework problems and review all the material discussed in class.

1. (a) Find the area of the triangle with vertices $(1,-1,1)$, $(1,2,3)$, $(2,-1,2)$.
   (b) Show that the line ℓ described by
   $$\frac{x+\sqrt{2}}{1} = \frac{y+3}{2\sqrt{2}} = \frac{z}{2}$$
   is parallel to the plane P that has equation $2\sqrt{2}x - 3y + 2\sqrt{2}z = 0$.
   (c) What is the distance D from ℓ to P?

2. Consider the line through the points $P := (1,2,3)$ and $Q := (2,0,1)$.
   (a) Find the intersection of the line with the xy plane.
   (b) Find the point on the line which is closest to the origin.
   (c) Find the distance from L to the origin.

3. An object moves according to the following equation
   $$\mathbf{r}(t) := 10\sin(t)\,\mathbf{i} + 8\cos(t)\,\mathbf{j} + (6+6\cos(t))\,\mathbf{k}$$
   (a) Assuming that the motion starts at $t=0$, determine how far the object has travelled when $t = 2\pi$.
   (b) Find the tangential and normal components a_T and a_N of the acceleration at $t = \pi/2$.

4. Consider the curve
   $$\mathbf{r}(t) := \cos(t)\,\mathbf{i} + \sin(t)\,\mathbf{j} - (\cos(t)+\sin(t))\,\mathbf{k}.$$
   Compute the velocity $\mathbf{v}$ and acceleration $\mathbf{a}$ and show that
   $$\frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = \mathbf{0}.$$

5. Find all critical points of the function
   $$f(x,y) := x^3 + y^3 - 6xy$$
   and characterize each one as local maximum, local minimum or saddle point.

6. Consider the function $f(x,y,z) := xye^z$.
   (a) Find the gradient of f.
   (b) Compute the directional derivative $D_{\mathbf{u}}f$ at the point $(1,1,0)$, where $\mathbf{u}$ is the unit vector in the direction $\mathbf{i} - \mathbf{j} + 2\mathbf{k}$.
   (c) Find a point on the level surface $f(x,y,z) = 1$ where the tangent plane is parallel to the plane $x + y + 2z = 3$.

7. Consider the surface S given by the equation
   $$x + y + z^2 + xy - x^2 - y^2 = 1.$$
   Find the equation of the tangent plane to S at the point $(1,0,-1)$.

8. Find the area in the first quadrant enclosed by the curves $y = x^3$, $y = 2x^3$ and $y = 8 - x^3$. **Hint:** You can use the change of variable $u = x^3$, $v = y + x^3$.

9. Compute the iterated integral
$$\int_0^3 \int_{y^2}^9 y \cos(x^2)\, dx\, dy$$
Hint: First reverse the order of integration.

10. Consider the vector field
$$\mathbf{F}(x, y, z) := x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$
and compute the flux integral
$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$
where S is the portion of the paraboloid $z = 9 - x^2 - y^2$ above the xy-plane and $\mathbf{n}$ is the unit normal vector directed upward.

11. Consider the vector field
$$\mathbf{F}(x, y, z) := e^x \cos(z)\,\mathbf{i} + y\,\mathbf{j} - e^x \sin z\,\mathbf{k}$$
   (a) Is $\mathbf{F}$ a conservative field? Justify your answer.
   (b) Evaluate
$$\int_C \mathbf{F} \cdot d\mathbf{r}$$
   where C is the curve parametrized by $\mathbf{r}(t) := e^{t^2(t^2-1)}\mathbf{i} + t\,\mathbf{j} + te^{t^{17}-1}\,k$ for $0 \le t \le 1$.

12. Let D denote the region consisting of the points (x, y, z) with $0 \le x \le 1$, $0 \le y \le 1$ and $0 \le z \le x$. Let Σ denote the boundary of D. Consider the vector field
$$\mathbf{F}(x, y, z) := (e^x + e^{yz})\,\mathbf{i} + (y^2 - z^2)\,\mathbf{j} + e^z\,\mathbf{k}.$$
Use the divergence theorem to evaluate the following integral
$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS,$$
where $\mathbf{n}$ is the normal to Σ that is directed outward from D.

13. Use Green's theorem to evaluate the integral
$$\int_C (x^2 + y^2 + 2y)\, dx + (3x + y^2)\, dy$$
where C is the circle $(x - 3)^2 + y^2 = 9$.

14. Consider the solid region D that lies above the xy plane and is bounded above by the sphere $x^2+y^2+z^2 = 4$, below by the sphere $x^2+y^2+z^2 = 1$ and on the sides by the cone $3z^2 = x^2+y^2$.
   (a) Compute the triple integral
$$I = \iiint_D (x^2 + y^2 + z^2)\, dV$$
   (b) Compute the flux integral
$$\iint_\Sigma \mathbf{F} \cdot \mathbf{n}\, dS$$
   where $\mathbf{F}$ is the vector field $\mathbf{F} := x^3\mathbf{i} + y^3\,\mathbf{j} + z^3\,\mathbf{k}$, Σ is the boundary of the region D and $\mathbf{n}$ is the outward normal vector on Σ. **Hint** You might find part (a) useful.