Math 241 **Midterm #1** **Fall 2012**

Solution

1. (a) Take $\mathbf{L} = \vec{AB} = (2, -3, -3)$. We get

$$x = 1 + 2t, \quad y = 3 - 3t, \quad z = 2 - 3t.$$

   (b) $P = (7/3, 1, 0)$ (when $z = 0$, we get $t = 2/3$ and so $x = 7/3$, $y = 1$).

2. (a) The vector $\mathbf{L} = (4, 2, 1)$ is parallel to the line and thus to $\mathcal{P}$.
   The point $P_1 = (-1, 0, 2)$ is on $\mathcal{P}$, so $\vec{P_0P_1} = (3, 1, 1)$ is parallel to $\mathcal{P}$.
   We deduce that

$$\mathbf{N} = \mathbf{L} \times \vec{P_0P_1} = \mathbf{i} - \mathbf{j} - 2\mathbf{k}$$

   is normal to $\mathcal{P}$.
   An equation for $\mathcal{P}$ is thus

$$(x + 4) - (y + 1) - 2(z - 1) = 0$$

   or

$$x - y - 2z = -5.$$

   (b) $d = \frac{|\mathbf{N} \cdot \vec{P_0P}|}{\|\mathbf{N}\|} = \frac{1}{\sqrt{6}}$.

3. We have $\mathbf{r}'(t) = 3\cos(3t)\mathbf{i} - 3\sin(3t)\mathbf{j} + 3t^{1/2}\mathbf{k}$ and so

$$\|\mathbf{r}'(t)\| = \sqrt{9\cos^2(3t) + 9\sin^2(3t) + 9t} = 3\sqrt{1+t}.$$

   We deduce

$$L = \int_0^2 \|\mathbf{r}'(t)\|\, dt = \int_0^2 3(1+t)^{1/2}\, dt = 3\int_1^3 u^{1/2} du = 2(3\sqrt{3} - 1).$$

4. (a) Since $\mathbf{v}(t) = \mathbf{r}'(t)$, we integrate $\mathbf{v}(t)$. Using the condition $\mathbf{r}(0) = \mathbf{i} + \mathbf{j}$, we find

$$\mathbf{r}(t) = (t^2 + 1)\mathbf{i} + (2t + 1)\mathbf{j} + \frac{1}{3}t^3\mathbf{k}$$

   (b) The acceleration is given by

$$\mathbf{a}(t) = \mathbf{v}'(t) = 2\mathbf{i} + 2t\mathbf{k}.$$

   The speed is $\|v(t)\| = \sqrt{4t^2 + 4 + t^4} = 2 + t^2$, so

$$a_T = \frac{d||\mathbf{v}(t)||}{dt} = 2t$$

   and $a_N^2 = \|\mathbf{a}\|^2 - a_T^2 = 4 + 4t^2 - 4t^2 = 4$, so

$$a_N = 2.$$

(c) We can use the formula $\kappa(t) = \frac{\|\mathbf{v}\times\mathbf{a}\|}{\|\mathbf{v}\|^3}$. We get:

$$\mathbf{v}\times\mathbf{a} = 4t\mathbf{i} - 2t^2\mathbf{j} - 4\mathbf{k}$$

and so

$$\|\mathbf{v}\times\mathbf{a}\| = \sqrt{16t^2 + 4t^4 + 16} = 2(2+t^2).$$

We deduce

$$\kappa(t) = \frac{2(2+t^2)}{(2+t^2)^3} = \frac{2}{(2+t^2)^2}.$$

5. We have $\mathbf{r}'(t) = t\cos t\mathbf{i} + t\sin t\mathbf{j}$ and $\|\mathbf{r}'(t)\| = \sqrt{t^2\cos^2 t + t^2\sin^2 t} = t$. So

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \cos t\mathbf{i} + \sin t\mathbf{j}.$$

Next, we find $\mathbf{T}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$ and $\|T'(t)\| = \sqrt{\sin^2 t + \cos^2 t} = 1$, so

$$\mathbf{N}(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}.$$