Math 241 **Midterm #2** **Fall 2012**

Solution

1. (a) We have $\frac{\partial f}{\partial x}(x,y) = 2e^{x^2-y^2} + 4x^2e^{x^2-y^2}$ and $\frac{\partial f}{\partial y}(x,y) = -4xye^{x^2-y^2}$, so

$$\frac{\partial f}{\partial x}(1,1) = 6 \quad \text{and} \quad \frac{\partial f}{\partial y}(1,1) = -4$$

(b) Using (a) and the fact that $f(1,1) = 2$, we get $z = 2 + 6(x-1) - 4(y-1)$, or

$$6x - 4y - z = 0.$$

(c)

$$\begin{aligned} f(1.02, 0.98) &\sim f(1,1) + \frac{\partial f}{\partial x}(1,1)(1.02-1) + \frac{\partial f}{\partial y}(1,1)(0.98-1) \\ &\sim 2 + 6\cdot 0.02 - 4\cdot(-0.02) \\ &\sim 2.2 \end{aligned}$$

2. (a) We have $\text{grad} f(x,y,z) = 2x\sin(yz)\mathbf{i} + x^2z\cos(yz)\mathbf{j} + x^2y\cos(yz)\mathbf{k}$ and so

$$\text{grad}\, f(1, \pi/2, 2) = -2\mathbf{j} - \frac{\pi}{2}\mathbf{k}.$$

Next we define $\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$. We deduce

$$D_{\mathbf{u}}f(1,\pi/2,2) = \text{grad}\, f(1,\pi/2,2)\cdot\mathbf{u} = -\frac{4}{3} + \frac{\pi}{3}.$$

(b)

$$\begin{aligned} \frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{1}{x-y}\frac{\partial x}{\partial s} - \frac{1}{x-y}\frac{\partial y}{\partial s} \\ &= \frac{1}{s^2+t^2-st}(2s-t) \end{aligned}$$

3. (a) Critical points satisfy $f_x = 3x^2 - 12x = 0$ and $f_y = -6y = 0$. We thus find two critical points:

$$(0,0) \text{ and } (4,0).$$

Furthermore, $D(x,y) = f_{xx}f_{yy} - f_{xy}^2 = -36(x-2)$, so
$D(0,0) > 0$ and $f_{xx}(0,0) = -12 < 0$ so $(0,0)$ is a relative maximum value
$D(4,0) < 0$: $(4,0)$ is a Saddle point.

(b) We must find all x, y such that

$$\begin{aligned} 2x &= 4\lambda x^3 \\ 2y &= 4\lambda y^3 \\ x^4 + y^4 &= 1 \end{aligned}$$

The first equation gives $x = 0$ or $x^2 = \frac{1}{2\lambda}$.
The Second equation gives $y = 0$ or $y^2 = \frac{1}{2\lambda}$.
If $x = 0$, then the last equation gives $y = \pm 1$.
If $y = 0$, then the last equation gives $x = \pm 1$.
If $x^2 = \frac{1}{2\lambda}$ and $y^2 = \frac{1}{2\lambda}$, then the last equation gives $(2\lambda)^2 = 2$ and so $x^2 = y^2 = \frac{1}{2\lambda} = \frac{\sqrt{2}}{2}$.
In conclusion, we found the following points:

$$(0, \pm 1),\ (\pm 1, 0),\ (\pm\sqrt{\frac{\sqrt{2}}{2}}, \pm\sqrt{\frac{\sqrt{2}}{2}})$$

Comparing the values of f at those points, we find that the maximum value of f is $\sqrt{2}$ and the minimum value of f is 1.

4.

$$\int_0^1 \int_0^y \cos(x)\, dxdy = \int_0^1 \sin(y) - \sin(0) dy = 1 - \cos(1)$$

5. The region R can be written as

$$R = \{(x, y)\,;\ -2 \leq x \leq 2, \quad x^2 \leq y \leq 4\}.$$

We thus have

$$\iint_R 3dA = \int_{-2}^2 \int_{x^2}^4 3\, dy\, dx = \int_{-2}^2 12 - 3x^2\, dx = 12x - x^3\Big]_{x=-2}^{x=2} = 32.$$