Solution to the review problems

1. $\dfrac{\partial z}{\partial x} = \dfrac{ye^{-y/x}}{x^3} - \dfrac{e^{y/x}}{x^2}$, $\dfrac{\partial z}{\partial y} = -\dfrac{e^{y/x}}{x^2}$, $\dfrac{\partial^2 z}{\partial y^2} = \dfrac{e^{y/x}}{x^3}$

2. Fixing $x = 0$ gives $f(0, y) = 0$, so $\lim_{(0,y)\to(0,0)} f(0, y) = 0$. Fixing $x = y$ gives $f(x, x) = \frac{1}{2}$, so $\lim_{(x,x)\to(0,0)} f(x, x) = \dfrac{1}{2}$. Since these are not equal, the limit does not exist.

3. $\mathrm{grad} f(x, y) = (10xy, 5x^2)$ and $\mathrm{grad} f(1, 1) = (10, 5)$. A unit vector in the direction of $\mathbf{i} - j$ is $\mathbf{u} = \frac{1}{\sqrt{2}}(1, -1)$, so $D_{\mathbf{u}} f(1, 1) = \frac{5}{\sqrt{2}}$.

4. (a) $\mathrm{grad} f(x, y) = (y \cos(xy), x \cos(xy))$, so the direction is $\mathrm{grad} f(1, \pi) = (-\pi, -1)$
   (b) $||\mathrm{grad} f(\pi, -1)|| = \sqrt{\pi^2 + 1}$

5. We have $\mathrm{grad} f(\pi/4, 4, 2) = (-6, -\pi/4, -\pi/4)$. The equation for the tangent plane is $-6x - \frac{\pi}{4}y - \frac{\pi}{4}z = -3\pi$

6. (a) $(-1, 0)$ is the only critical point
   (b) $D(-1, 0) = -17 < 0$ so f has a saddle point at $(-1, 0)$

7.
   - f has one critical point at $(1, 0)$ and $f(1, 0) = 0$.
   - On the boundary $x = 0$, we have $f = 1 + y^2$ which has minimum value 1 and maximum value 5.
   - On the boundary, $x^2 + y^2 = 4$, we can write $y^2 = 4 - x^2$ and so $f = -2x + 5$. Hence f has maximum value 5 and minimum value 1.

   In conclusion, f has maximum value 5 (attained at the points $(0, \pm 2)$ and minimum value 0 (attained at $(1, 0)$).

8. f has a maximum of $3\sqrt{3}$ attained at the point $(\sqrt{3}, 1)$ and a minimum of $-3\sqrt{3}$ attained at the point $(-\sqrt{3}, 1)$

9. $\displaystyle\int_0^1 \int_{-3}^3 \frac{xy}{x^2 + 1}\, dydx = 9 \ln 2$

10. $\displaystyle\int_0^1 \int_0^{x^2} x \cos y \, dy \, dx = (1 - \cos 1)/2$

11. Switching the order of integration gives $\displaystyle\int_0^1 \int_y^1 \sin(x^2)\, dx\, dy = \int_0^1 \int_0^x \sin(x^2)\, dy\, dx = (1 - \cos 1)/2$