Solution

1. The domain D is given (using spherical coordinate) by

$$D = \{(\theta, \phi, \rho)\,;\, 0 \le \theta \le 2\pi,\ 0 \le \phi \le \frac{\pi}{4},\ 0 \le \rho \le 2\}$$

so

$$\begin{aligned}\iiint_D 5z^2\,dV &= \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 5(\rho\cos\phi)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \int_0^{2\pi}\int_0^{\pi/4} 2^5(\cos\phi)^2\ \sin\phi\,d\phi\,d\theta \\ &= -\int_0^{2\pi}\int_1^{\sqrt{2}/2} 2^5(u)^2\,du\,d\theta \\ &= \frac{2^6\pi}{3}(1-\frac{\sqrt{2}}{4})\end{aligned}$$

2.

$$\begin{aligned}\int_{\mathcal{C}} y\,dx - x\,dy + xy\,dz &= \int_0^{\pi/2} \sin t(-\sin t)\,dt - \cos t(\cos t)\,dt + \cos t\sin t\,dt \\ &= \int_0^{\pi/2} -1+\cos t\sin t\,dt \\ &= -t + \frac{1}{2}\sin^2(t)\Big]_0^{\pi/2} \\ &= -\frac{\pi}{2}+\frac{1}{2}\end{aligned}$$

3. We have $M = (y+1)e^{x^2}$ and $N = yx^2+y^2$, so

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2xy - e^{x^2}.$$

Using Green's theorem, we get

$$\int_{\mathcal{C}} (y+1)e^{x^2}\,dx + (yx^2+y^2)\,dy = \iint_R 2xy - e^{x^2}\,dA$$

where $R = \{(x,y)\,;\, 0 \le x \le 1,\ 0 \le y \le x\}$. So

$$\begin{aligned}\int_{\mathcal{C}} (y+1)e^{x^2}\,dx + (yx^2+y^2)\,dy &= \int_0^1\int_0^x 2xy - e^{x^2}\,dy\,dx \\ &= \int_0^1 x^3 - xe^{x^2}\,dx \\ &= \frac{1}{2}\int_0^1 u - e^u\,du \qquad (\text{with } u = x^2) \\ &= \frac{3}{4} - \frac{e}{2}\end{aligned}$$

4. The surface Σ is the graph of $f(x,y) = 16 - x^2 - y^2$ for $(x,y) \in R$ where

$$R = \{(x,y)\,;\, x^2 + y^2 \leq 16\}.$$

We have $f_x = -2x,\ f_y = -2y$ and so (since the normal vector points upward)

$$\begin{aligned}
\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_R -f_x M - f_y N + P\,dA \\
&= \iint_R \frac{2x^2}{\sqrt{x^2+y^2}} + \frac{2y^2}{\sqrt{x^2+y^2}} + 4\,dA \\
&= \iint_R 2\sqrt{x^2+y^2} + 4\,dA
\end{aligned}$$

Using polar coordinates, we deduce

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \int_0^{2\pi}\int_0^4 (2r+4)r\,dr\,d\theta = \frac{7\cdot 64}{3}\pi$$

5. The divergence theorem yields

$$\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \mathrm{div}\mathbf{F}\,dV.$$

We first compute

$$\mathrm{div}\mathbf{F} = 1 + 1 + 0 = 2.$$

In cylindrical coordinates, the cylinder has equation $r = 2\cos\theta,\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$, so the region D can be described as

$$D = \{(\theta, r, z)\,;\, -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2},\ 0 \leq r \leq 2\cos\theta,\ 0 \leq z \leq 4 - r^2\}.$$

We deduce

$$\begin{aligned}
\iint_\Sigma \mathbf{F}\cdot\mathbf{n}\,dS &= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta}\int_0^{4-r^2} 2\,r\,dz\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} 8r - 2r^3\,dr\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 4r^2 - \frac{1}{2}r^4\Big]_0^{2\cos\theta}\,d\theta \\
&= \int_{-\pi/2}^{\pi/2} 16\cos^2(\theta) - 8\cos^4(\theta)\,d\theta \\
&= 8\theta + 4\sin(2\theta) - 3\theta - 2\sin(2\theta) - \frac{1}{4}\sin(4\theta)\Big]_{-\pi/2}^{\pi/2} \\
&= 5\pi
\end{aligned}$$