Review problems for the third midterm

These are typical problems that could be on the exam, though there are more problems here than could possibly be on the exam (you can expect 4 or 5 problems on the exam).

The exam may include material that is not on the review and the wording of questions may be somewhat different. Make sure that you do all suggested homework problems and review all the material discussed in class.

1. Consider the solid region D that is bounded above by the paraboloid $z = 7 - x^2 - y^2$, below by the xy plane, and on the sides by the cylinder $x^2 + y^2 = 4$. We define the triple integral

$$I = \int\int\int_D (x^2 + y^2)^{3/2} \, dV.$$

   (a) Express I as an iterated integral in cylindrical coordinates.
   (b) Evaluate I by using your answer to part (a).

2. Let D be the solid region below the plane $z = x + y + 5$, inside the cylinder $x^2 + y^2 = 1$, outside the cylinder $x^2 + y^2 = x$ in the first octant. Express the volume of D as an iterated integral in cylindrical coordinates.

3. Using the change of coordinates $x = u + 17v$ and $y = 17u$, set up an integral in u and v to evaluate

$$\int\int_R x^\pi \, dA$$

   where R is the region bounded by $y = \frac{17}{18}x$, $y = 17x$ and $y = 17$. You do not need to evaluate the integral.

4. Consider the vector field $\vec{F}$ defined by

$$\vec{F}(x, y, z) = 2xyz\vec{i} + x^2z\vec{j} + (x^2y + 1)\vec{k}.$$

   (a) Describe the domain of $\vec{F}$. Show that $\vec{F}$ is irrotational, i.e. show that $\vec{curl}\vec{F} = \vec{0}$ for all points (x, y, z) in the domain of $\vec{F}$.
   (b) Determine a function $f(x, y, z)$ such that $\vec{F}(x, y, z) = \nabla f(x, y, z)$.
   (c) Evaluate the line integral

$$\int_C \vec{F} \cdot d\vec{r},$$

   where $\vec{F}$ is given above and C is the curve parametrized by

$$\vec{r}(t) = \cos(\pi t^7)\vec{i} + t^7\vec{j} + \frac{2t}{1 + t^2}\vec{k}, \qquad -1 \leq t \leq 1$$

   where the initial point of C corresponds to $t = -1$ while the terminal point corresponds to $t = 1$.

5. Let C be the curve parametrized by the vector function

$$\vec{r}(t) = t\vec{i} + t^2\vec{j} + t^3\vec{k} \qquad \text{for } 0 \leq t \leq 1.$$

   Evaluate the line integral $I = \int_C xy\,dx + zx\,dy - xyz\,dz$

6. For $1 \le u \le 3$, $0 \le v \le 2\pi$, the function

$$\mathbf{r}(u,v) = 4u\cos v\mathbf{i} + 4u\sin v\mathbf{j} + \ln u\mathbf{k}$$

parametrizes a surface Σ. Find the tangent plane to Σ at $\mathbf{r}(2,\pi)$.

7. Consider the line integral

$$I = \int_C y(x^2+y^2)\,dx - x(x^2+y^2)\,dy,$$

where C is the circle $x^2+y^2=1$ oriented counterclockwise on the xy-plane.

(a) Use Greens theorem to write I as a double integral over a suitable region R of the xy-plane. Define the region R clearly.

(b) Evaluate the integral I, first using your result of part (a), then by direct computation (make sure that you get the same answer).

8. An oriented surface Σ is the lower half of an ellipsoid. This surface is described by $z = -2\sqrt{1-x^2-y^2}$ and has a unit normal vector $\vec{n}$ directed upward. Consider the flux integral

$$I = \int\int_\Sigma \vec{F}\cdot\vec{n}\,dS$$

where $\vec{F} = \vec{i} + \vec{j} + 2\vec{k}$.

(a) Express the flux integral I as a double integral over a suitable region R of the xy plane. Describe R.

(b) Evaluate I.

9. Consider the vector field

$$\vec{F}(x,y,z) = x^3\vec{i} + y^3\vec{j} + z^3\vec{k}$$

(a) By use of Gauss's Theorem (also known as the Divergence Theorem), evaluate the flux integral

$$I = \int\int_\Sigma \vec{F}\cdot\vec{n}\,dS$$

where Σ is the boundary of the cube D with vertices at $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$, $(0,0,1)$, $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$, and $\vec{n}$ is the unit normal directed outward from D.

(b) Compute directly the flux integral I of part (a). You should show that the result agrees with that found in part (a).