Math 461

Final - Practice exam

This is a practice exam. The actual exam may include material that is not on this practice exam and the wording of questions may be different. Make sure that you do all suggested homework problems and review all the material discussed in class.

1. (a) The matrices

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ 11 & 12 & 13 & 14 & 15 \\ 16 & 17 & 18 & 19 & 20 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -1 & -2 & -3 \\ 0 & 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

are row equivalent.

i. Find a basis for the column space of A.

ii. Find a basis for the null space of A.

iii. Compute:

- Rank (A)
- dim (Nul(A))

(b) Show that the vectors $\vec{u_1} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$, $\vec{u_2} = \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix}$ and $\vec{u_3} = \begin{bmatrix} -7 \\ 1 \\ 4 \end{bmatrix}$ are linearly dependent and find a linear dependence relation among them.

2. The matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$.

(a) Find an **orthogonal** basis of eigenvectors for the matrix A.

(b) Give the matrices P and D which **orthogonally** diagonalize A.

3. (a) Given $A = \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ 0 & 1 \\ 1 & -2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 1 \\ -8 \\ 2 \end{bmatrix}$, find a least squares solution of $A\vec{x} = \vec{b}$.

(b) [10pts] A certain experiment produces the data $(1, 3.2)$, $(2, 5.1)$ and $(3, 2.4)$. Give the design matrix X and the observation vector $\vec{y}$ that leads to a least-squares fit of these points by a function of the form

$$y = A\cos x + B\sin x$$

(you do not need to solve the corresponding system).

4. Compute the determinant of the following matrix:

$$A = \begin{bmatrix} 1 & 3 & -2 & -1 \\ -1 & 7 & 2 & 2 \\ 0 & 3 & 0 & 0 \\ 8 & 5 & -4 & 5 \end{bmatrix}$$

5. Consider the set

$$W = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ; x_1,\ x_2,\ x_3 \text{ in } \mathbb{R} \text{ such that } x_1 - 2x_2 + 5x_3 = 0 \right\}.$$

   (a) Show that W is a subspace of $\mathbb{R}^3$.

   (b) Find a basis for W.

   (c) Find an orthogonal basis for W.

   (d) Find the distance from $\vec{b} = \begin{bmatrix} 1 \\ 3 \\ 7 \end{bmatrix}$ to W.

6. (a) Find the **complex** eigenvalues and eigenvectors of the following matrix:

$$B = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$$

   (b) Consider the quadratic form $Q(\vec{x}) = 2x_1^2 - x_2^2 + 4x_1x_2$. Find the symmetric matrix A such that $Q(\vec{x}) = \vec{x}^T A\vec{x}$ and classify the quadratic form Q as positive definite, negative definite or indefinite.

7. We recall that $\mathbb{P}_2$ denotes the vector space of all polynomials of degree less than or equal to 2 and that $\mathcal{E} = \{1, t, t^2\}$ is a basis of $\mathbb{P}_2$.

   We define

$$\vec{p_1}(t) = 1 + 2t, \quad \vec{p_2}(t) = t - t^2, \quad \vec{p_3}(t) = t + t^2.$$

   (a) Show that $\mathcal{B} = \{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is a basis for $\mathbb{P}_2$.

   (b) Let $\vec{q}(t) = 1 + 3t + t^2$. Compute $[\vec{q}(t)]_{\mathcal{E}}$ and $[\vec{q}(t)]_{\mathcal{B}}$.

   (c) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the transformation $T(\vec{p}(t)) = \vec{p}\,'(t) - \vec{p}(t)$ (where $\vec{p}\,'(t)$ denotes the derivative of the polynomial $\vec{p}(t)$). Find the matrix of T relative to the basis $\mathcal{B}$.