Math 461

Final - Practice exam

Solution

1. (a) i. $\left\{ \begin{bmatrix} 1 \\ 6 \\ 11 \\ 16 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 12 \\ 17 \end{bmatrix} \right\}$

   ii. $\left\{ \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

   iii.
   - $\text{Rank}(A) = 2$
   - $\dim\ (\text{Nul}(A)) = 3$

   (b) $-4\vec{u}_1 + 3\vec{u}_2 + 2\vec{u}_3 = \vec{0}$

2. (a) $\lambda = 1$: We first get $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$. But $\vec{u}_1 \cdot \vec{u}_2 \neq 0$, so we use G.S.: $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/2 \\ 1/2 \\ 1 \end{bmatrix} \right\}$.

   $\lambda = 4$: $\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$.

   (b) $P = \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{6} & 1/\sqrt{3} \\ 1/\sqrt{2} & 1/\sqrt{6} & -1\sqrt{3} \\ 0 & 2/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$ $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

3. (a) $\vec{x_0} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$.

   (b) $X = \begin{bmatrix} \cos 1 & \sin 1 \\ \cos 2 & \sin 2 \\ \cos 3 & \sin 3 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 3.2 \\ 5.1 \\ 2.4 \end{bmatrix}$

4. $\det(A) = 36$.

5. (a) $W = \text{Nul}(A)$ where $A = [1\ -2\ 5]$.

   (b) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

   (c) $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} \right\}$.

(d) $d = \sqrt{30}$

6. (a) $\lambda = \frac{3}{2} \pm i\frac{\sqrt{3}}{2}$, $\vec{v} = \begin{bmatrix} 1 \\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \end{bmatrix}$

(b) $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, $\lambda = 3,\ -2$. Indefinite.

7. (a) $[\vec{p_1}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $[\vec{p_2}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, $[\vec{p_3}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \sim I_3$, so $\mathcal{B} = \{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is a basis for $\mathbb{P}_2$.

(b) $[\vec{q}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, $[\vec{q}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

(c) $T(\vec{p_1}) = 1 - 2t$, $T(\vec{p_2}) = 1 - 3t + t^2$, $T(\vec{p_3}) = 1 + t - t^2$.

$[T(\vec{p_1})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}$, $[T(\vec{p_2})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}$, $[T(\vec{p_3})]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

So

$$[T]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 1 \\ -2 & -3 & 0 \\ -2 & -2 & -1 \end{bmatrix}$$