Math 461 - Linear Algebra for Scientists and Engineers

MATLAB Assignment #3

The rank of a matrix: If A is a matrix, the command `rank(A)` computes the rank of A. Because of roundoff errors, it is difficult to compute the rank (if an entry is 10^{-16}, is it a pivot or should it be zero?). For example, the command

```
>> rank([1 0 ; 0 .0000000000000000000001])
```

will return 1 for the rank since it figures your matrix is close enough to the rank 1 matrix $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. It is possible to defeat this sophistication, but we won't need to for this assignment.

If you generate a random $m \times n$ matrix, it will always have rank as big as possible (the minimum of m and n). Try this by typing `rank(rand(4,6))`, `rank(rand(7,3))`, etc. There is a way of generating random matrices with smaller rank by using the following fact:

Fact: *Suppose $A = BC$ where B is a $m \times k$ matrix of rank k and C is a $k \times n$ matrix of rank k. Then A is a $m \times n$ matrix of rank k.*

(You can prove this fact by showing that Nul(BC)=Nul(C)).

For instance, the command

```
>> rand(4,2)*rand(2,4)
```

produces a random 4×4 matrix with rank 2 (check it!).

The Null space: The matlab command `null(A)` will produce a matrix whose columns are a basis for the null space of A. It will not be the one you get by row reducing, because it will be an orthonormal basis (we will learn about these later). Matlab also attempts to account for roundoff error when using the command `null`.

Problem 1. Determine whether $\vec{w}$ is in the column space of A, the null space of A, or both, where

$$\vec{w} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \end{bmatrix} \quad A = \begin{bmatrix} 7 & 6 & -4 & 1 \\ -5 & -1 & 0 & -2 \\ 9 & -11 & 7 & -3 \\ 19 & -9 & 7 & 1 \end{bmatrix}$$

Problem 2. Generate a random 6×8 matrix with rank 3. Check that it has rank 3 by row reducing the matrix.

Ask matlab for a basis of the null space. We want to check that it is indeed a basis of the null space: Check that the vectors are indeed in the null space. Check that they are linearly independent. Why do we know that they must span the null space?

Problem 3. Find a basis for the space spanned by the given vectors:

$$\begin{bmatrix} 8 \\ 9 \\ -3 \\ -6 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 1 \\ -4 \\ 4 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ -9 \\ 6 \\ -7 \end{bmatrix}, \begin{bmatrix} 6 \\ 8 \\ 4 \\ -7 \\ 10 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \\ 11 \\ -8 \\ -7 \end{bmatrix}.$$

Problem 4. Show that $\{t, \sin t, \cos 2t, \sin t \cos t\}$ is a linearly independent set of functions defined on $\mathbb{R}$. Start by assuming that

$$c_1 t + c_2 \sin t + c_3 \cos 2t + c_4 \sin t \cos t = 0$$

for all t. Choose specific values of t (for instance $t = 0, 1, 2 \ldots$) until you get a system with enough equations to determine that all the c_i must be 0.

Problem 5. Determine whether the following set of polynomials forms a basis of $\mathbb{P}_3(\mathbb{R})$.

$$p_1 = 3 + 7t,\ p_2 = 5 + t - 2t^3,\ p_3 = t - 2t^2,\ p_4 = 1 + 16t - 6t^2 + 2t^3.$$

Problem 6. Consider the matrix

$$Q = \begin{bmatrix} .90 & .01 & .09 \\ .01 & .90 & .01 \\ .09 & .09 & .90 \end{bmatrix}$$

1. Compute Q^k for $k = 10, 20, \ldots, 100$. What do you observe?
2. What does it imply for the behavior of the discrete dynamical system $\vec{x}_{k+1} = Q\vec{x}_k$ when k goes to infinity?
3. Find a vector $\vec{p}$ such that

   $$Q\vec{p} = \vec{p}$$

   (such a vector is called a steady-state vector).