Spring 2013 - Math 461
Midterm #1 - February 18

- Write your name and section number **on every page**. (You only need to write and sign the honor pledge on the first page).
- This exam has **4 questions**. Do each question on a **different** answer sheet.
- Show enough work to justify your answers. **Unjustified answers will receive no credit.**
- **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

1. (a) [10pts] True or False. **Justify** your answer carefully. **Unjustified answers will receive no credit.**
   i. If the augmented matrix $[A\ \vec{b}]$ has a pivot in every column, then the equation $A\vec{x} = \vec{b}$ has exactly one solution.
   ii. For any linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$, there exists a 2×3 matrix A such that $T(\vec{x}) = A\vec{x}$ for all $\vec{x}$.
   iii. The solution set of an equation $A\vec{x} = \vec{b}$ can always be expressed as $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ for some vectors $\vec{v}_1, \ldots, \vec{v}_p$.
   iv. 6 vectors in $\mathbb{R}^4$ are always linearly dependent.

   (b) [15pts] Consider the matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -3 & -6 & -1 & 0 \\ 1 & 2 & 1 & -2 \end{bmatrix}$ and the vector $\vec{b} = \begin{bmatrix} -4 \\ 7 \\ 1 \end{bmatrix}$.

   You are given that the augmented matrix $[A\ \ \vec{b}]$ is row equivalent to the matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & -4 \\ 0 & 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   i. Find the solution set of the homogeneous matrix equation $A\vec{x} = \vec{0}$. **Write your answer in parametric vector form.**
   ii. Find the solution set of the matrix equation $A\vec{x} = \vec{b}$. **Write your answer in parametric vector form.**

2. [25pts] Consider the following vectors in $\mathbb{R}^4$:

$$\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \\ -2 \\ -1 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \\ 2 \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} -2 \\ 1 \\ 4 \\ 0 \end{bmatrix}, \quad \vec{b} = \begin{bmatrix} -3 \\ 3 \\ 1 \\ h \end{bmatrix}$$

   where h is a real parameter.

   (a) Are the vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ linearly independent. (Justify your answer).
   (b) For what value(s) of h (if any) is $\vec{b}$ in $\text{Span}\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$? (Justify your answer)

3. [20pts] Let $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, $\vec{v}_4$ be four vectors in $\mathbb{R}^4$ such that the matrix $[\vec{v}_1 \;\; \vec{v}_2 \;\; \vec{v}_3 \;\; \vec{v}_4]$ is row equivalent to the matrix

$$\begin{bmatrix} 1 & 0 & -4 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Do the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ span $\mathbb{R}^4$? (Justify your answer)

(b) Explain why the vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ are linearly dependent and find a linear dependence relation.

4. [30pts]

(a) Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a linear transformation such that

$$T(\vec{x}) = \begin{bmatrix} 2x_1 - 3x_2 \\ -x_1 + 5x_2 \\ x_1 + x_2 \end{bmatrix}$$

where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

Show that T is a linear transformation and find its standard matrix.

Is T one-to-one? Is T onto $\mathbb{R}^3$?

(b) Let S be the linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$ which transforms $\vec{e_1}$ into $\begin{bmatrix} 1 \\ -3 \end{bmatrix}$ and $\vec{e_2}$ into $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ (where $\vec{e_1}$ and $\vec{e_2}$ denote the columns of the identity matrix I_2).

Find the standard matrices of the transformation S and of its inverse S^{-1}.