Spring 2013 - Math 461

Midterm #1 - Solutions

1. (a) [10pts] **Unjustified answers will receive no credit.**
   i. False (such a system will be inconsistent).
   ii. True (theorem seen in class).
   iii. False (only if $\vec{b} = \vec{0}$).
   iv. True (the corresponding matrix has at most 4 pivots, so it cannot have a pivot in every columns).

   (b) [15pts]
   i. The homogeneous equation has solution

$$\begin{cases} x_1 = -2x_2 - x_4 \\ x_2 \text{ is free} \\ x_3 = 3x_4 \\ x_4 \text{ is free} \end{cases}$$

   so the solution are given by $\vec{x} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ with x_2, x_4 in $\mathbb{R}$.

   ii. The non-homogeneous equation has solution

$$\begin{cases} x_1 = -4 - 2x_2 - x_4 \\ x_2 \text{ is free} \\ x_3 = 5 + 3x_4 \\ x_4 \text{ is free} \end{cases}$$

   so the solution are given by $\vec{x} = \begin{bmatrix} -4 \\ 0 \\ 5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$ with x_2, x_4 in $\mathbb{R}$.

2. [25pts] We first row reduce:

$$[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3 \quad \vec{b}] \sim \begin{bmatrix} 1 & 3 & -2 & -3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & h+4 \end{bmatrix}$$

   (a) $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ is linearly independent (pivot in all columns of $[\vec{a}_1 \quad \vec{a}_2 \quad \vec{a}_3]$).

   (b) The equation $x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 = \vec{b}$ is consistent iff the matrix above does not have a pivot in the last column. So $\vec{b}$ is in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ iff $h = -4$.

3. [20pts]

   (a) No (no pivot in the last row).

(b) The vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ are linearly dependent because the matrix above has no pivot in the third column. The equation $x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4 = \vec{0}$ is equivalent to the system

$$\begin{cases} x_1 \quad -4x_3 = 0 \\ x_2 \quad +2x_3 = 0 \\ x_4 = 0 \end{cases}$$

which has a free variable (x_3). Taking (for instance) $x_3 = 1$, we find $x_1 = 4$, $x_2 = -2$, $x_4 = 0$, which gives the linear dependence relation

$$4\vec{v}_1 - 2\vec{v}_2 + \vec{v}_3 + 0\vec{v}_4 = \vec{0}$$

4. [30pts]

(a) We write

$$T(\vec{x}) = x_1 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -3 \\ -1 & 5 \\ 1 & 1 \end{bmatrix} \vec{x}$$

So T is a matrix transformation with matrix $A = \begin{bmatrix} 2 & -3 \\ -1 & 5 \\ 1 & 1 \end{bmatrix}$. It is therefore a linear transformation.

The columns of A are linearly independent (two vectors not multiple of each other), so T is one-to-one.

The columns of A do not span $\mathbb{R}^3$ (A cannot have a pivot in every row), so T is not onto.

(b) The standard matrix of S is $B = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix}$.

The standard matrix of S^{-1} is $B^{-1} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$.