Math 461

Midterm #1 - Practice exam

This is a practice exam. The actual exam may include material that is not on this practice exam and the wording of questions may be different. Make sure that you do all suggested homework problems and review all the material discussed in class.

1. (a) [10pts] True or False. Be sure to **justify** your responses. No justification, no credit...
   i. Two linear systems are equivalent if they have the same solution set.
   ii. The equation $A\vec{x} = \vec{b}$ has at least one solution if and only if $\vec{b}$ is a linear combination of the columns of A.
   iii. The equation $A\vec{x} = \vec{b}$ is consistent if the augmented matrix $[A \quad \vec{b}]$ has a pivot in every row.
   iv. The columns of any 4×5 matrix are linearly dependent.

   (b) [10pts] Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by this succession of transformations:
   - Rotate counter-clockwise through an angle of $\frac{\pi}{2}$ radians (90°).
   - Then reflect through the x_2-axis.

   Find the standard matrix of the transformation T.

2. Let A be the following matrix:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 3 & 0 \\ 2 & -2 & 2 \end{bmatrix}$$

   (a) [20pts] Let $\vec{b} = \begin{bmatrix} -3 \\ -5 \\ 2 \end{bmatrix}$. Does the equation $A\vec{x} = \vec{b}$ have a solution? If not, then explain why. If so, then find the solution(s). Express your answer in vector parametric form.

   (b) [5pts] Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\vec{x}) = A\vec{x}$ (with the same matrix A as above). Is T one-to-one? Is T onto?

3. Consider the following three vectors in $\mathbb{R}^3$:

$$\vec{a_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad \vec{a_2} = \begin{bmatrix} 3 \\ -5 \\ 1 \end{bmatrix} \quad \vec{a_3} = \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix},$$

and let

$$\vec{b} = \begin{bmatrix} -2 \\ 8 \\ h \end{bmatrix}$$

where h is a real parameter.

(a) [8pts] For what value(s) of h is $\vec{b}$ in Span$\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$

(b) [7pts] For each value of h found in part (a), express $\vec{b}$ as a linear combination of the vectors $\vec{a_1}, \vec{a_2}$ and $\vec{a_3}$.

(c) [8pts] Are the vectors $\vec{a}_1, \vec{a}_2, \vec{a}_3$ linearly independent? If not, find a linear dependence relation.

(d) [7pts] Does the set of vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ span $\mathbb{R}^3$? Are there any values of h for which the set of vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b}\}$ spans $\mathbb{R}^3$?

4. Let $S : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$S(\vec{e_1}) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} \qquad S(\vec{e_2}) = \begin{bmatrix} -2 \\ -3 \\ 2 \end{bmatrix} \qquad S(\vec{e_3}) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

where $\vec{e_1}$, $\vec{e_2}$, $\vec{e_3}$ denote the columns of the identity matrix in $\mathbb{R}^3$.

(a) [5pts] Find the standard matrix A of the transformation S.

(b) [5pts] What is the image of the vector $\vec{c} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ under S?

(c) [15pts] Is A invertible? If so, find the inverse of A.