Math 461

Midterm #1 - Solution to the practice exam

1. (a)
   i. True (definition)
   ii. True: The equation $A\vec{x} = \vec{b}$ is equivalent to the vector equation $x_1\vec{a_1} + \cdots + x_n\vec{a_n} = \vec{b}$.
   iii. False: The equation $A\vec{x} = \vec{b}$ is consistent if and only if the augmented matrix $[A \quad \vec{b}]$ does not have a pivot in the last column
   iv. True: A 4×5 matrix has at most 4 pivots and thus cannot have a pivot in every column.

   (b) $T(\vec{e_1}) = \vec{e_2}$, $T(\vec{e_2}) = \vec{e_1}$ so $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. (a) $[A\ \vec{b}] \sim \begin{bmatrix} 1 & 0 & 3/2 & -1 \\ 0 & 1 & 1/2 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has no pivot in the last column, so the system is consistent, and the solutions are given by

$$\vec{x} = \begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3/2 \\ -1/2 \\ 1 \end{bmatrix}$$

   (b) [5pts] A does not have a pivot in the last column, so T is not one-to-one.
   A does not have a pivot in the last row, so A is not onto.

3. $[\vec{a_1} \quad \vec{a_2} \quad \vec{a_3} \quad \vec{b}] \sim \begin{bmatrix} 1 & 3 & -1 & -2 \\ 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & h+3 \end{bmatrix}$

   (a) $h = -3$ (system is consistent)

   (b) $\vec{b} = 7\vec{a_1} - 3\vec{a_2} + 0\vec{a_3}$.

   (c) $[\vec{a_1} \quad \vec{a_2} \quad \vec{a_3}] \sim \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$ has no pivot in the last column, so the vectors are linearly dependent, and

$$-5\vec{a}_1 + 2\vec{a}_2 + \vec{a}_3 = \vec{0}$$

   (d) $[\vec{a_1} \quad \vec{a_2} \quad \vec{a_3}]$ has no pivot in the last row, so the set of vectors $\{\vec{a}_1, \vec{a}_2, \vec{a}_3\}$ does not span $\mathbb{R}^3$.
   $\{\vec{a}_1, \vec{a}_2, \vec{a}_3, \vec{b}\}$ spans $\mathbb{R}^3$ for all $h \neq -3$.

4. (a) $A = \begin{bmatrix} 1 & -2 & 1 \\ 3 & -3 & 0 \\ 0 & 2 & -1 \end{bmatrix}$

(b) $S(\vec{c}) = A\vec{c} = \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$

(c) $A^{-1} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1/3 & 1 \\ 2 & -2/3 & 1 \end{bmatrix}$