Spring 2013 - Math 461

Midterm #2 - April 1st

- Write your name and section number **on every page**. (You only need to write and sign the honor pledge on the first page).
- This exam has **4 questions**. Do each question on a **different** answer sheet.
- Show enough work to justify your answers. **Unjustified answers will receive no credit.**
- **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

1. (a) [10pts] True or False. **Justify** your answers carefully.
   i. If A is a 3×3 matrix whose columns span $\mathbb{R}^3$, then $\det(A) \neq 0$.
   ii. The solution set of an equation of the form $A\vec{x} = \vec{b}$ is always a vector space.
   iii. $\mathrm{Nul}(A)$ is the kernel of the linear transformation $\vec{x} \mapsto A\vec{x}$.
   iv. If A is not invertible, then 0 is an eigenvalue of A.

   (b) [15pts] The matrix $A = \begin{bmatrix} 1 & 2 & 0 & 1 \\ -3 & -6 & -1 & 0 \\ 1 & 2 & 1 & -2 \end{bmatrix}$ is row equivalent to the matrix $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
   i. What is the Rank of A? What is the dimension of the null space of A?
   ii. Find a basis for the column space of A.
   iii. Find a basis for the null space of A.

2. (a) [15pts] Let $A = \begin{bmatrix} 1 & 5 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -1 & 3 \\ 5 & 7 & 1 & 3 & 4 \\ 1 & -2 & 0 & 4 & 1 \end{bmatrix}$
   i. Compute $\det(A)$ and $\det(-2A)$.
   ii. Explain why A is invertible and compute $\det(A^{-1})$.

   (b) [10pts] Find the value(s) of t for which the following vectors are linearly dependent:

$$\vec{a}_1 = \begin{bmatrix} 0 \\ t \\ -2 \end{bmatrix}, \quad \vec{a}_2 = \begin{bmatrix} 5 \\ 2 \\ t \end{bmatrix}, \quad \vec{a}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

3. (a) [10pts] Let H be the set of all polynomials of the form $\vec{p} = a + (a+b)t + bt^2$ for a and b in $\mathbb{R}$. Show that H is subspace of $P_2(\mathbb{R})$ and find a spanning set for H.

   (b) [15pts] Let $A = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 4 & 1 \\ -1 & -1 & 2 \end{bmatrix}$. Show that $\lambda = 3$ is a eigenvalue for A and find a basis of the corresponding eigenspace (that is find a basis of $\mathrm{Nul}(A - 3I)$).

4. (a) [15pts] Let $A = \begin{bmatrix} 1 & -6 \\ 2 & -6 \end{bmatrix}$

   i. Show that the vectors $\vec{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$ are two eigenvectors for A. In each case give the corresponding eigenvalue.

   ii. Let $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$.
   Give $\mathcal{P}_{\mathcal{E} \leftarrow \mathcal{B}}$ (the change of coordinate matrix from $\mathcal{B}$ to the standard basis of $\mathbb{R}^2$) and $\mathcal{P}_{\mathcal{B} \leftarrow \mathcal{E}}$ (the change of coordinate matrix from the standard basis of $\mathbb{R}^2$ to $\mathcal{B}$).

   iii. Let $\vec{x} = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Find $[\vec{x}]_{\mathcal{B}}$.

(b) [10pts] We recall that $P_2(\mathbb{R})$ is the vector space of all polynomials with degree less than or equal to 2. We also recall that $\mathcal{B} = \{1,\ t\ , t^2\}$ is a basis for $P_2(\mathbb{R})$. Let

$$\vec{p_1} = 1 - 2t + t^2, \quad \vec{p_2} = 3 - 5t + 4t^2, \quad \vec{p_3} = 2t + 3t^2.$$

Show that $\mathcal{C} = \{\vec{p_1},\ \vec{p_2},\ \vec{p_3}\}$ is a basis for $P_2(\mathbb{R})$.