Spring 2013 - Math 461

Midterm #2 - Solution

1. (a)
   i. True (such a matrix is invertible).
   ii. False (only if $\vec{b} = \vec{0}$).
   iii. True (definition of the kernel of the linear transformation).
   iv. True (if A is not invertible, then $A\vec{x} = \vec{0}$ has a non trivial solution, so 0 is an eigenvalue of A).

   (b)
   i. Rank$(A) = 2$ (number of pivot columns). dim(Nul$(A)) = 2$ (number of free variables).
   ii. $\left\{ \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$
   iii. $\vec{x} = \begin{bmatrix} -2x_2 - x_4 \\ x_2 \\ 3x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, so $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$ is a basis for the null space of A.

2. (a)
   i. $\det(A) = 11$ and $\det(-2A) = (-2)^5 11$.
   ii. $\det(A) \neq 0$, so A is invertible and $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{11}$.

   (b) The vectors are linearly dependent iff the matrix $[\vec{a}_1\ \vec{a}_2\ \vec{a}_3]$ is not invertible, that is iff its determinant is zero:

   $$\det([\vec{a}_1\ \vec{a}_2\ \vec{a}_3]) = t^2 - 5t + 4 = (t-1)(t-4) = 0$$

   So the vectors are linearly dependent iff $t = 1$ or $t = 4$.

3. (a) We have $\vec{p} = a + (a+b)t + bt^2 = a(1+t) + b(t+t^2)$, so

   $$H = \text{Span}\{1+t, t+t^2\}$$

   in particular, H is a subspace since it is the span of some set of vectors.

   (b) We have $A - 3I = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

   In particular, the equation $A\vec{x} = 3\vec{x}$ has non trivial solutions, so 3 is an eigenvalue. The vectors in Nul$(A - 3I)$ are of the form

   $$\vec{x} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

   so $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis of Nul$(A - 3I)$.

4. (a) i. We have $A\vec{v}_1 = -3\vec{v}_1$ and $A\vec{v}_2 = -2\vec{v}_2$. So $\vec{v}_1$ is a e-vector with e-value -3 and $\vec{v}_2$ is a e-vector with e-value -2.

ii. $\mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}} = [\vec{v}_1\ \vec{v}_2] = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$, $\mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}} = \mathcal{P}_{\mathcal{E}\leftarrow\mathcal{B}}{}^{-1} = \begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$

iii. $[\vec{x}]_{\mathcal{B}} = \mathcal{P}_{\mathcal{B}\leftarrow\mathcal{E}}\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

(b) We have $[\vec{p_1}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $[\vec{p_2}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}$, $[\vec{p_3}]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, and

$$\begin{bmatrix} 1 & 3 & 0 \\ -2 & -5 & 2 \\ 1 & 4 & 3 \end{bmatrix} \sim I_3$$

so $\{\vec{p_1}, \vec{p_2}, \vec{p_3}\}$ is linearly independent. Since $\dim(P_2(\mathbb{R})) = 3$, it is a basis.