Math 461

Midterm #2 - Practice exam

This is a practice exam. The actual exam may include material that is not on this practice exam and the wording of questions may be different. Make sure that you do all suggested homework problems and review all the material discussed in class.

1. (a) True or False. Justify carefully your answer.
   i. The set H of all vectors in $\mathbb{R}^3$ whose last entry is zero is a subspace of $\mathbb{R}^3$.
   ii. The set S of all polynomials of the form $p(t) = 1 + at + bt^2$ with a and b in $\mathbb{R}$ is a subspace of $P_2(\mathbb{R})$.
   iii. If A is a 5×7 matrix of Rank 3, then $\mathrm{Nul}(A)$ is a subspace of dimension 2.
   iv. If A is row equivalent to B, then $\det A = \det B$.

   (b) The following matrices are row equivalent:

$$A = \begin{bmatrix} 1 & -1 & -7 & 1 & 10 \\ 2 & 0 & -8 & -1 & 6 \\ -1 & 1 & 7 & 0 & -6 \\ -1 & 0 & 4 & 1 & -1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & -4 & 0 & 5 \\ 0 & 1 & 3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

   i. Find a basis for $\mathrm{Col}(A)$
   ii. Find a basis for $\mathrm{Nul}(A)$
   iii. Find a basis for $\mathrm{Row}(A)$

2. (a) Compute the determinant of the matrices A, B and C, where:

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 \\ -1 & 0 & -5 & 1 \\ 3 & 1 & 4 & -2 \\ 1 & 0 & 0 & 2 \end{bmatrix}, \qquad B = 2A, \qquad C = A^4.$$

   (b) Find the value(s) of the parameter t for which the following system has a unique solution

$$\begin{cases} 3t\,x_1 + 2x_2 = 5 \\ 9x_1 + 2t\,x_2 = -2 \end{cases}$$

3. (a) Let

$$H = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} ;\ x_1,\ x_2,\ x_3 \text{ in } \mathbb{R} \text{ such that } x_1 + 2x_3 = 0 \text{ and } x_2 - 5x_3 = 0 \right\}.$$

   i. Explain why H is a subspace of $\mathbb{R}^3$.
   ii. Find a basis for H. What is the dimension of H?

(b) Let G be the set of vectors of the form $\begin{bmatrix} a+2c+d \\ -4a+3b-5c-d \\ 3a-b+5c+2d \end{bmatrix}$ with a, b, c, d in $\mathbb{R}$.

i. Explain why G is a subspace of $\mathbb{R}^3$.
ii. Find a basis for G. What is the dimension of G?

4. (a) Let

$$\vec{b_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \qquad \vec{b_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

i. Explain why $\mathcal{B} = \{\vec{b_1}, \vec{b_2}\}$ is a basis of $\mathbb{R}^2$.
ii. Find the coordinate vector of $\vec{u} = \begin{bmatrix} 8 \\ -1 \end{bmatrix}$ relative to the basis $\mathcal{B}$.

(b) We recall that $P_2(\mathbb{R})$ is the vector space of all polynomials with degree less than or equal to 2. We also recall that $\mathcal{B} = \{1,\, t\,, t^2\}$ is a basis for $P_2(\mathbb{R})$.
Let

$$\vec{p_1} = t^2, \quad \vec{p_2} = -1+t, \quad \vec{p_3} = 1+2t.$$

i. What is the dimension of $P_2(\mathbb{R})$?
ii. Show that $\mathcal{C} = \{\vec{p_1},\, \vec{p_2},\, \vec{p_3}\}$ is a basis for $P_2(\mathbb{R})$.
iii. Let $\vec{p} = -6-6t+3t^2$. Find $[\vec{p}]_{\mathcal{C}}$.

5. (a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}$$

(b) The following matrix:

$$A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$$

has characteristic equation $(10-\lambda)(1-\lambda)^2 = 0$ (you do not need to check this fact). Find a basis for each eigenspace of A.