Spring 2013 - Math 461
Midterm #3 - Solutions

1. (a) [10pts]
   i. True (theorem)
   ii. False (only if the column of the matrix are linearly independent)
   iii. True (theorem)
   iv. False (only similar matrices have the same eigenvalues - for instance any invertible matrix is row equivalent to I, and yet may have eigenvalues not equal to 1).

   (b) [15pts]
   i. $\vec{x}_1 = A\vec{x}_0 = PDP^{-1}\vec{x}_0 = PD\begin{bmatrix} -2 \\ 1 \end{bmatrix} = P\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -5 \\ -5 \end{bmatrix}$.
   ii. The matrix A has eigenvectors $\vec{u}_1$ and $\vec{u}_2$ (with corresponding eigenvalues .5 and 2), and $\vec{x}_0 = -2\vec{u}_1 + \vec{u}_2$, so

$$\vec{x}_k = -2(.5)^k\vec{u}_1 + 2^k\vec{u}_2.$$

2. [25pts]

   (a) The eigenvalues of an upper triangular matrix are the diagonal entries (theorem), so we have $\lambda_1 = 2$ and $\lambda_2 = -1$ (with multiplicity 2).
   (b) For $\lambda_1 = 2$, we find

$$A - 2I = \begin{bmatrix} 0 & -3 & 3 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

so the corresponding eigenspace is $\text{Span}\left\{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right\}$

For $\lambda_2 = -1$, we find

$$A + I = \begin{bmatrix} 3 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

so the corresponding eigenspace is $\text{Span}\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$.

We deduce that $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

3. (a) [15pts] $\det(A - \lambda I) = (\lambda - 2)^2 + 5$, so $\lambda_1 = 2 + i\sqrt{5}$ and $\lambda_2 = 2 - i\sqrt{5}$.
For λ_1, we find

$$A - \lambda_1 I = \begin{bmatrix} -1 - i\sqrt{5} & 2 \\ -3 & 1 - i\sqrt{5} \end{bmatrix}$$

and keeping only the first row, we get $(-1 - i\sqrt{5})x_1 + 2x_2 = 0$. If we take $x_1 = 2$, we find $x_2 = 1 + i\sqrt{5}$. So

$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 + i\sqrt{5} \end{bmatrix}$$

For λ_2, we can take $\vec{v}_2 = \overline{\vec{v}_1} = \begin{bmatrix} 2 \\ 1 - i\sqrt{5} \end{bmatrix}$

(b) [10pts] $A = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$.
$\det(A - \lambda I) = \lambda^2 - 4\lambda - 1 = (\lambda - 2)^2 - 5$, so A has eigenvalues $\lambda_1 = 2 + \sqrt{5} > 0$ and $\lambda_2 = 2 - \sqrt{5} < 0$, and thus Q is indefinite.

4. [25pts]

(a) [10pts] Using the Gram-Schmidt process, we find:

$$\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = \vec{u}_2 - \frac{\vec{u}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \vec{u}_2 - 2\vec{v}_1 = \begin{bmatrix} 2 \\ -2 \\ -3 \end{bmatrix}$$

(b) [10pts]

$$\text{Proj}_W \vec{b} = \frac{\vec{b} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{b} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 = 2\vec{v}_1 - \vec{v}_2 = \begin{bmatrix} 0 \\ -2 \\ 7 \end{bmatrix}$$

(c) [5pts] The distance from $\vec{b}$ to W is $\|\vec{b} - \text{Proj}_W \vec{b}\| = \left\| \begin{bmatrix} 10 \\ 7 \\ 2 \end{bmatrix} \right\| = \sqrt{153}$