Math 461

Midterm #3 - Practice exam

This is a practice exam. The actual exam may include material that is not on this practice exam and the wording of questions may be different. Make sure that you do all suggested homework problems and review all the material discussed in class.

1. True or False. Be sure to justify your answers.
   (a) If 0 is an eigenvalue for A, then A is not diagonalizable.
   (b) If an $n \times n$ matrix A is diagonalizable, then A has n distinct eigenvalues.
   (c) Similar matrices have exactly the same eigenvalues.
   (d) Any solution of $A^T A\vec{x} = A^T\vec{b}$ is a least-squares solution of $A\vec{x} = \vec{b}$.

2. Consider the matrix
$$A = \begin{bmatrix} 2 & -1 & 3 \\ 0 & 0 & 6 \\ 0 & 3 & -7 \end{bmatrix}.$$
   (a) Check that $\vec{v}_1 = \begin{bmatrix} -1 \\ -2 \\ 3 \end{bmatrix}$ is an eigenvector for A and find the corresponding eigenvalue.
   (b) Check that $\lambda = 2$ is an eigenvalue of A and find a basis of eigenvectors for the corresponding eigenspace.
   (c) Diagonalize A (use your answers to (a) and (b)).

3. Let $A = \begin{bmatrix} 3 & 1 \\ -7 & -1 \end{bmatrix}$.
   (a) Find the (complex) eigenvalues and eigenvectors of A.
   (b) Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$.
   (c) The transformation $T : \vec{x} \mapsto C\vec{x}$ (with the matrix C from (b)) is the composition of a rotation and a scaling. Give the angle φ of the rotation and the scale factor r.

4. Consider the quadratic form $Q(\vec{x}) = 2x_1^2 - x_2^2 + 4x_1x_2$. Find the symmetric matrix A such that $Q(\vec{x}) = \vec{x}^T A\vec{x}$ and classify the quadratic form Q as positive definite, negative definite or indefinite.

5. Find the least-squares solution of $A\vec{x} = \vec{b}$ when

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 3 \\ 1 \\ -4 \\ 2 \end{bmatrix}$$

6. Let

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 0 & 2 & -1 \\ 1 & -2 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Use the Gram-Schmidt process to find an **orthogonal** basis for $W = \mathrm{Col}(A)$.

7. Let $\vec{u} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and define $L = \mathrm{Span}\{\vec{u}\}$.

(a) Find the orthogonal projection of $\vec{x}$ onto L.

(b) Find the orthogonal projection of $\vec{x}$ onto $W = L^{\perp}$.