Fall 2009 - Math 463 Section 0201
Complex Variables for Scientists and Engineers
Homework #3 - Due Thursday September 24th **in class**

1. Sketch the region onto which the set of points satisfying $1 \le \mathrm{Re}(z) \le 2$, $1 \le \mathrm{Im}(z) \le 2$ is mapped by the transformation $f(z) = e^z$.

2. We examine some properties of the exponential function
   (a) Are there any complex number z such that $e^z = 0$? (justify your answer)
   (b) Prove that $e^{\overline{z}} = \overline{e^z}$.
   (c) What can be said about z if $|e^{-z}| < 1$?

3. For each of the following, find the image of S under the transformation $w = f(z)$.
   (a) $f(z) = iz$; S is the circle $|z - 1| = 2$.
   (b) $f(z) = (1 + i)z$; S is the line $y = 2x + 1$.
   (c) $f(z) = 1/z$; S is the circle $|z| = 2$.

4. Use parametrization to find the image of the circle $|z - z_0| = R$ under the transformation $f(z) = iz - 2$.

5. Show that the image of the vertical line $\mathrm{Re}(z) = 1$ under the transformation $f(z) = 1/z$ is a circle of radius $1/2$, centered at $z_0 = 1/2$.

6. Use the properties of limits to compute the following
   (a) $\displaystyle\lim_{z \to 2-i} (z^2 - z)$
   (b) $\displaystyle\lim_{z \to 1+i} \frac{z - \overline{z}}{z + \overline{z}}$
   (c) $\displaystyle\lim_{z \to 2+i} \frac{z^2 - (2+i)^2}{z - (2+i)}$
   (d) $\displaystyle\lim_{z \to e^{i\pi/4}} \left(z + \frac{1}{z}\right)$

7. Consider the limit $\displaystyle\lim_{z \to 0} \left(\frac{\overline{z}}{z}\right)^2$
   (a) What value does the limit approach as z approaches 0 along the real axis?
   (b) What value does the limit approach as z approaches 0 along the imaginary axis?
   (c) Does (a) and (b) implies that $\displaystyle\lim_{z \to 0} \left(\frac{\overline{z}}{z}\right)^2$ exists? Explain.
   (d) What value does the limit approach as z approaches 0 along the line $y = x$? What can you now say about $\displaystyle\lim_{z \to 0} \left(\frac{\overline{z}}{z}\right)^2$.

8. Compute the following limits

   (a) $\displaystyle\lim_{z\to\infty} \frac{z^2 + iz - 2}{(1+2i)z^2}$

   (b) $\displaystyle\lim_{z\to i} \frac{z^2 - 1}{z^2 + 1}$

9. For each of the following, show that the function is continuous at the given point

   (a) $f(z) = z^3 - \frac{1}{z}$; $z_0 = 3i$

   (b) $f(z) = \begin{cases} \frac{z^3-1}{z-1}, & |z| \neq 1 \\ 3, & |z| = 1 \end{cases}$; $z_0 = 1$

10. Show that the function $f(z) = \text{Arg}(z)$ is discontinuous at $z = -1$.