Fall 2009 - Math 463 Section 0201
Complex Variables for Scientists and Engineers
Homework #5 - Due Thursday October 15th **in class**

1. Find the domain in which the following functions are analytic and compute their derivatives:
   (a) $f(z) = z^2 e^{z+1}$
   (b) $f(z) = ie^{1/z}$
   (c) $f(z) = (z+1)\text{Log}z$
   (d) $f(z) = \text{Log}(z^2)$
   (e) $f(z) = \sin(iz+4)$

2. (a) Give the real part and imaginary part of $\text{Log}(-2-2i)$.
   (b) Give the real part and imaginary part of $\text{Log}(-ei)$.
   (c) Find all complex values of $\log(-\sqrt{3}+i)$.
   (d) Find all the values of z such that $\log z = \ln 3 + i\frac{\pi}{4}$.
   (e) Find all the values of z such that $\log z = 1 + i\pi$.

3. Using the definition of the log, show that $\log(z_1 z_2) = \log(z_1) + \log(z_2)$ for all z_1, z_2 not zero.

4. Let S be the set of nonzero complex numbers in the first quadrant (that is $z \neq 0$ and $0 \leq \text{Arg}(z) \leq \pi/2$).
   (a) Describe and sketch the image of the set S by the transformation $f(z) = \text{Log}(z)$.
   (b) Describe and sketch the image of the set S by the transformation $f(z) = \log(z)$.

5. Find the principal values of
   (a) $(1+i)^i$
   (b) i^{1+i}
   (c) $(-1+i\sqrt{3})^{3/2}$

6. Using the properties of the complex exponential function, show the following properties of the complex trigonometric functions:
   (a) Show that sin and cos are entire and that
   $$\frac{d}{dz}\sin(z) = \cos(z) \quad \text{and} \quad \frac{d}{dz}\cos(z) = -\sin(z)$$
   (b) Show that $\cos(-z) = \cos(z)$ and $\sin(-z) = -\sin(z)$

(c) Show that cos and sin are 2π periodic (that is $\cos(z + 2\pi) = \cos(z)$).

7. Find the real part and imaginary part of

   (a) $\sin(4i)$

   (b) $\cos(2 - 4i)$

   (c) $\sin^{-1}(\sqrt{2})$

8. Find all complex values z satisfying

$$\cos z = \sin z$$

9. **[Optional]**

   (a) Using Euler Formula, show that

$$e^{iz_1} e^{iz_2} = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 + i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2)$$

   (b) Use the fact that

$$\sin(z_1 + z_2) = \frac{1}{2i}(e^{iz_1} e^{iz_2} - e^{-iz_1} e^{-iz_2})$$

   show that

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2.$$

   (c) Differentiating both side of this last equality with respect to z_1, show that

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2.$$

   (d) Using this last equality, show that

$$\cos^2 z + \sin^2 z = 1 \qquad \text{for all } z.$$