MATH 410 – Spring 2009 – Test 1 – M. Brin

(1) Find $\lim_{n\to\infty} n^{205}\left(\sqrt{n^{410}+2}-\sqrt{n^{410}-1}\right)$.

Multiply the numerator and denominator by $\left(\sqrt{n^{410}+2}+\sqrt{n^{410}-1}\right)$

$n^{205}\left(\sqrt{n^{410}+2}-\sqrt{n^{410}-1}\right) = \dfrac{3n^{205}}{\sqrt{n^{410}+2}+\sqrt{n^{410}-1}} = \dfrac{3}{\sqrt{1+\frac{2}{n^{410}}}+\sqrt{1-\frac{1}{n^{410}}}} \to$

$\dfrac{3}{2}$.

(2) Suppose $f\colon \mathbb{R}\to\mathbb{R}$ is a strictly increasing function defined for all real numbers. Prove that the limit from the right $\lim_{x\to 0+} f(x)$ exists.

Let $S=\{f(x)\colon x>0\}$. The set S is not empty (contains $f(1)$) and bounded from below (for example, by $f(0)$). Therefore, by the completeness axiom, $\alpha=\inf S$ exists. Claim: $\lim_{x\to 0^+} f(x)=\alpha$. To prove this fix any $\epsilon>0$. Since $\alpha+\epsilon$ is not a lower bound fro S, there is $\delta>0$ such that $f(delta)<\alpha+\epsilon$. Since f is increasing, $f(x)<f(\delta)<\alpha+\epsilon$ for each x with $0<x<\delta$. Hence, $0\le f(x)-\alpha<\epsilon$ for $0<x<\delta$.

(3) A subset $S\subset\mathbb{R}$ is open if for every $x\in S$ there is $\epsilon>0$ such that $(x-\epsilon,x+\epsilon)\subset S$.

Let $S_1, S_2, \dots, S_k$ be open subsets of $\mathbb{R}$. Prove that the intersection $\bigcap_{i=1}^{k} S_i$ is open.

Let $x\in\bigcap_{i=1}^k S_i$. Then $x\in S_i$ for each i and there is $\epsilon_i>0$ such that $(x-\epsilon_i,x+\epsilon_i)\subset S_i$. Choose $\epsilon=\min_{i=1,2,\dots,k}\epsilon_i$. Then the interval $(x-\epsilon,x+\epsilon)$ is contained in each S_i, and hence in the intersection.

(4) Suppose that $-x^4\le f(x)\le x^4$ for all $x\in\mathbb{R}$.

Prove that f is differentiable at 0 and that $f'(0)=0$.

Observe that $f(0)=0$. We have: $\left|\dfrac{f(x)-f(0)}{x-0}\right| \le \left|\frac{f(x)}{x}\right| \le \frac{|x^4|}{|x|} = |x|^3 \to 0$ as $x\to 0$.

Hence $f'(0)=0$.

(5) Suppose that a function $g\colon\mathbb{R}\to\mathbb{R}$ is differentiable at $x=0$. Also, suppose that $g(1/n)=0$ for each natural number n.

Prove that a) $g(0)=0$, b) $g'(0)=0$.

a) Since g is differentiable at 0, it is continuous at 0. Therefore, since $\lim_{n\to\infty} 1/n=0$, we have $f(0)=\lim_{n\to\infty} g(1/n)=\lim_{n\to\infty} 0=0$.

b) Since $g'(0)$ exists, $\lim_{n\to\infty}\dfrac{g(x_n)-g(0)}{x-0}=g'(0)$ for every sequence $\{x_n\}$ with $\lim_{n\to\infty} x_n = 0$. Observe that $\dfrac{g(1/n)-g(0)}{1/n-0}=0$ for each n. Hence, $\lim_{n\to\infty}\dfrac{g(1/n)-g(0)}{1/n-0}=0$ and $g'(0)=0$.