MATH 410 – Spring 2009 – Test 2 – M. Brin

(1) Prove that the equation $f(x) = x^5 + 410x + 1 = 0$ has exactly one real solution.
Solution. Observe that $f(-1) < 0$ and $f(+1) > 0$. Therefore, by the Intermediate Value Theorem (polynomials are continuous), there is at least one root. On the other hand, $f'(x) = 5x^4 + 410 > 0$ for all x. Hence f is strictly increasing and cannot take on the same value at two different points.

(2) Suppose that $P(x)$ is a polynomial of degree at most 410 and suppose that $P(x_0) = P'(x_0) = \cdots = P^{(410)}(x_0) = 0$ for some x_0.
Prove that $P(x) \equiv 0$.
Solution. Rewrite P as a polynomial in $(x - x_0)$:
$P(x) = a_{410}(x - x_0)^{410} + a_{409}(x - x_0)^{409} + \cdots + a_1(x - x_0) + a_0$. Observe that $(P(x_0))^k = a_k \cdot k!$ for $0 \le k \le n$. Therefore all coefficients a_k are 0 and P is 0.

(3) Assume that a (finite) partition P of an interval $[a, b]$ is a refinement of a partition Q (i.e., each partition point of Q is a partition point of P). Let f be a bounded function. Prove that $U(f, P) \le U(f, Q)$ (the Refinement Lemma).
Solution. Consider an interval $I = [x_i, x_{i+1}]$ of Q. This interval contributes the term $f_{max}(x_{i+1} - x_i)$ to the upper sum $U(f, Q)$, where f_{max} is the supremum of f on I. Let $y_0 = x_i, y_1, y_2, \ldots, y_k = x_{i+1}$ be the partition points of P on I. Let f_j be the supremum of f on $[y_{j-1}, y_j]$. Then the total contribution from I to $U(f, P)$ is $\sum_{j=1}^{k} f_j(y_j - y_{j-1}) \le \sum_{j=1}^{k} f_{max}(y_j - y_{j-1}) = f_{max}(x_{i+1} - x_i)$.

(4) Prove that for all $x > 0$

$$1 + \frac{x}{3} - \frac{x^2}{9} < (1 + x)^{1/3} < 1 + \frac{x}{3}.$$

Solution. Let $f(x) = (1 + x)^{1/3}$. Then $f(0) = 1$, $f'(x) = \frac{1}{3}(1 + x)^{-2/3}$, $f''(x) = \frac{2}{9}(1 + x)^{-5/3}$, $f'(0) = \frac{1}{3}$. Therefore (the Taylor polynomial of degree 1 at 0 with Lagrange remainder):

$$f(x) = 1 + \frac{1}{3}x - \frac{1}{9}x^2(1 + c)^{-5/3} \text{ for some } 0 < c < x.$$

The left inequality follows since the third term is negative. The right inequality follows since $\frac{1}{9}(1 + c)^{-5/3} < \frac{1}{9}$.

(5) Suppose that a function $F\colon \mathbb{R} \to \mathbb{R}$ has derivatives of all orders and that
i) $F'(x) - 410F(x) = 0$ for all x, ii) $F(0) = 1$.
a) Find a formula for the coefficients of the nth Taylor polynomial for F at $x_0 = 0$.
b) Prove that the Taylor expansion converges at every point x.
Solution. a) Since $F'(x) = 410F(x)$, we have $F^{(n)}(x) = 410F^{(n-1)}(x)$ and $F^{(n)}(0) = 410^n$. The coefficient at x^n is $\frac{410^n}{n!}$.
b) For an arbitrary x, let M be the supremum of F on $[0, x]$. Then $F^{(n)}(c) \le M \cdot 410^n$. Hence, by Theorem 8.14, the Taylor expansion converges.