Math 341 M. Brin Spring 2010 EXAM 1

Each problem is 20 points. Solve each problem on a separate answer sheet. Show all work. Justify and mark clearly all answers, unless instructed otherwise.

1. Compute the flux of the field $F(x, y, z) = (x^2\mathbf{i} + xy\mathbf{j} - 2xz\mathbf{k})$ through the surface of the tetrahedron with vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ (the normal $\mathbf{n}$ is directed outward).

Solution. $\mathbf{div}\, F = 2x + x - 2x = x$; by the divergence theorem the flux equals the integral

$$\int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} dz\, x = 1/24$$

2. Compute the flux of the field $G(x, y, z) = (yz\mathbf{i} + xy\mathbf{j} + xz\mathbf{k})$ through the boundary of the solid region inside the cylinder $x^2 + z^2 = 1$ and between the planes $y = -1$ and $y = 1$ (the normal $\mathbf{n}$ is directed outward).

Solution. $\mathbf{div}\, G = x + x = 2x$; by the divergence theorem the flux equals the integral

$$\int_{-1}^1 dy \iint_{x^2+z^2\le 1} 2x\, dS = \int_{-1}^1 dy \int_0^{2\pi} d\theta \int_0^1 dr\, r \cdot 2r\cos\theta = 0$$

3. Let $H = \big(x^2\mathbf{i} + (y^2 \sin z)\mathbf{j} - z^2(e^x - \tan y)\mathbf{k}\big)$. Find the flux of $\mathbf{curl}\, H$ through the hemisphere $z = \sqrt{1 - x^2 - y^2}$ (the normal $\mathbf{n}$ is directed upward).

Solution. By the Stokes Theorem, replace the hemisphere with the disk $x^2 + y^2 \le 1$, $z = 0$. The third (vertical, or z, or $\mathbf{k}$) component of $\mathbf{curl}\, H$ is 0. Hence $\mathbf{curl}\, H \cdot \mathbf{n} = 0$, and the flux equals 0.

4. Let $V = \big((x^2+z)\mathbf{i} + (y^2+x)\mathbf{j} + (z^2+y)\mathbf{k}\big)$. Compute the circulation integral $\oint_C V \cdot d\mathbf{x}$, where C is the intersection of the sphere $x^2+y^2+z^2 = 1$ and the cone $z = \sqrt{x^2+y^2}$; the curve C is oriented counterclockwise as seen from above.

Solution. The curve C is the circle $x^2 + y^2 = 1/2$, $z = 1/\sqrt{2}$. It bounds a disk of radius $1/\sqrt{2}$ whose normal $\mathbf{n}$ is vertical. The tird component of $\mathbf{curl}\, V$ is 1. Hence $\mathbf{curl}\, V \cdot \mathbf{n} = 1$. By the Stokes Theorem, the circulation equals the flux of the curl, which is equal to the area of the disk $\pi/2$.

5. Solve the initial value problem

$$\frac{dx}{dt} = \frac{3t^2+4t+2}{2(x-1)}, \quad x(0) = -1\,.$$

Solution. $\int 2(x-1)\,dx = \int (3t^2+4t+2)\,dt$; $x^2 - 2x = t^3 + 2t^2 + 2t + C$. From the initial condition $C = 3$. $x(t) = 1 \pm \sqrt{1+t^3+2t^2+2t+3}$; and using the initial condition again

$$x(t) = 1 - \sqrt{t^3+2t^2+2t+4}$$