Math 341 M. Brin Spring 2010 EXAM 2

1. A 100-gallon tank initially contains 50 gallons of water with a total of 10 pounds of salt dissloved in it. A drain is opened in the bottom which lets out 1 gallon of solution per minute. Simultaneously, salt solution begins to be added at 2 gallons per minute with a concentration of 2 pounds per gallon.
Find the concentration of salt in the tank at the moment it becomes full.

Solution. Let $V(t)$ be the volume of solution in the tank and $s(t)$ the amount of salt in the tank. Then $V(t) = 50 + t$, $s(0) = 10$ and $s'(t) = 4 - \frac{s}{50+t}$. $(s(50+t))' = 200 + 4t$; $s(50+t) = 200t + 2t^2 + C$; $C = 500$; $s(50) = 155$; $s(50)/100 = 1.55$.

2. For the differential equation $y'' - 2y' + 10y = 0$
a) find the general solution (in real form),
b) find the solution such that $y(0) = 2$, $y'(0) = 5$ (in real form).

Solution. a) $r = 1 \pm 3i$, $y = e^x(C_1 \cos 3x + C_2 \sin 3x)$.
b) Since $y(0) = 2$ we have $C_1 = 2$. Substitute $y = e^x(2\cos 3x + C_2 \sin 3x)$ in the equation to get $C_2 = 1$. $y = e^x(2\cos 3x + \sin 3x)$.

3. a) Find the general solution of $y'' - 3y' + 2y = 0$.
b) Find an appropriate form for a solution of
$y'' - 3y' + 2y = 3x^2 \sin x - 7x \cos x + 4xe^{-x} - x^2 e^{-2x}$

Solution. a) $r^2 - 3r + 2 = 0$, $r = 1, 2$, $y = C_1 e^x + C_2 e^{2x}$.
b) $y = (A_1x^2 + A_2x + A_3)\cos x + (B_1x^2 + B_2x + B_3)\sin x + (C_1x + C_2)e^{-x} + (D_1x^2 + D_2x + D_3)e^{-2x}$.

4. a) Find the general solution of $x'' + 2x' + 2x = 0$ (in real form).
b) Find the steady state of $x'' + 2x' + 2x = \cos t$ in the form $A\cos(\omega t - \alpha)$.

Solution. a) $r^2 + 2r + 2 = 0$; $r = -1 \pm i$; $x(t) = e^{-t}(C_1 \cos t + C_2 \sin t)$
b) Substitute $y = a\cos t + b\sin t$ into the equation to get $\frac{1}{5}(\cos t + 2\sin t)$; $y = \frac{1}{\sqrt{5}}\cos(t - \alpha)$, where $\cos\alpha = \frac{1}{\sqrt{5}}$, $\sin\alpha = 2\frac{1}{\sqrt{5}}$.

5. a) Find the general solution of the system
$\dot{x} = 8x - 3y$, $\dot{y} = 10x - 3y$.
b) Find the particular solution satisfying $x(0) = 0$, $y(0) = 1$.

Solution. a) $\lambda^2 - 5\lambda + 6 = 0$, $\lambda = 2, 3$. For $\lambda = 2$, an eigenvector is $\binom{1}{2}$. For $\lambda = 3$, an eigenvector is $\binom{3}{5}$. $\binom{x}{y} = C_1 e^{2t}\binom{1}{2} + C_2 e^{3t}\binom{3}{5}$.
b) The initial conditions give $C_1 = 3$, $C_2 = -1$; $\binom{x}{y} = 3e^{2t}\binom{1}{2} - e^{3t}\binom{3}{5}$.