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INTERSECTION NUMBERS OF CYCLES ON LOCALLY SYMMETRIC SPACES AND FOURIER COEFFICIENTS OF HOLOMORPHIC MODULAR FORMS IN SEVERAL COMPLEX VARIABLES

by STEPHEN S. KUDLA* *and* JOHN J. MILLSON**

Abstract. — Using the theta correspondence we construct liftings from the cohomology with compact supports of locally symmetric spaces associated to $O(p, q)$ (resp. $U(p, q)$) of degree nq (resp. Hodge type nq, nq) to the space of classical holomorphic Siegel modular forms of weight $(p + q)/2$ and genus n (resp. holomorphic hermitian modular forms of weight $p + q$ and genus n). It is important to note that the cohomology with compact supports contains the cuspidal harmonic forms by Borel [3]. We can express the Fourier coefficients of the lift of η in terms of periods of η over certain totally geodesic cycles—generalizing Shintani's solution [21] of a conjecture of Shimura. We then choose η to be the Poincaré dual of a (finite) cycle and obtain a collection of formulas analogous to those of Hirzebruch-Zagier [8]. In our previous work we constructed the above lifting but we were unable to prove that it took values in the *holomorphic* forms. Moreover, we were unable to compute the indefinite Fourier coefficients of a lifted class. By Koecher's Theorem we may now conclude that all such coefficients are zero.

This paper is the result of a number of years work [9-14], and was announced in [17]. For some time we have been studying the relationship between two types of cohomology classes for arithmetic quotients of the symmetric spaces attached to orthogonal and unitary groups. The first type of cohomology class has a geometric description as the Poincaré dual classes to natural cycles on the above arithmetic quotients. These cycles are themselves unions of arithmetic quotients of totally geodesic subsymmetric spaces associated to smaller orthogonal or unitary groups. They generalize the classical Hurwitz correspondences and the cycles in the Hilbert modular surfaces considered by Hirzebruch-Zagier [8]. The second type of cohomology class has an analytic description in terms of automorphic forms on the above arithmetic quotients constructed using the theta correspondence. This correspondence is realized by an integral transformation with kernel a theta function defined on a product of two locally symmetric spaces. The new development that led to this paper is the discovery of a new method to use the Cauchy-

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Riemann equations in the general theory of the theta correspondence. Our method is based on a study of the double complex of relative Lie algebra cohomology with values in the oscillator representation associated to a dual reductive pair. We use such algebraic considerations to show that the $\bar{\partial}$-operator applied to one variable of the kernel is an exact differential form in the other variable. Combining this result with Stokes' Theorem, we are able to deduce that the transforms of *closed* rapidly decreasing differential forms are *holomorphic* Siegel or hermitian modular forms. Our idea may be summarized by the following formal calculation which is made precise in Lemma 3.3. All the integrals are with respect to z.

$$\begin{aligned}\bar{\partial}\left(\int_{\mathrm{M}}\theta_{\varphi}(\tau, z) \wedge \eta(z)\right) &= \int_{\mathrm{M}}\bar{\partial}\theta_{\varphi}(\tau, z) \wedge \eta(z) = \int_{\mathrm{M}}\theta_{\bar{\partial}\varphi}(\tau, z) \wedge \eta(z)\\ &= \int_{\mathrm{M}}\theta_{d\psi}(\tau, z) \wedge \eta(z) = \int_{\mathrm{M}} d(\theta_{\psi}(\tau, z) \wedge \eta(z)) = 0.\end{aligned}$$

The key step $\bar{\partial}\varphi = d\psi$ is the double complex argument alluded to above and the last equality is Stokes' Theorem—here it is necessary that η is rapidly decreasing. We should perhaps point out that the Howe correspondence on the centers of the universal enveloping algebras of G and G′ (see below) gives only that the above integral is harmonic, i.e., annihilated by $\square = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ and this does not imply that the integral is annihilated by $\bar{\partial}$ unless it is in L^2 which is not usually the case. The harmonicity of the integral does not imply the vanishing theorem for its indefinite Fourier coefficients. This vanishing is of critical importance to us. Moreover the correspondence at infinity underlying our correspondence does not always agree with Howe's quotient correspondence. For example if $\mathrm{G} = \mathrm{O}(2p, 2q)$ then our correspondence assigns a *holomorphic* representation of $\mathrm{Sp}_n(\mathbf{R})$ to the trivial representation of $\mathrm{O}(2p, 2q)$. We will discuss this point in detail in a later work. Our lift should be of significance in arithmetic algebraic geometry since it assigns algebraic objects (holomorphic modular forms) to transcendental objects (cuspidal harmonic forms). In particular if it can be established that our lift is injective one could assign compatible systems of ℓ-adic Galois representations to the above cuspidal harmonic forms. In order to state our first theorem we will need to establish some notation.

Let M be a quotient of the symmetric space D of $\mathrm{G} = \mathrm{O}(p, q)$ (resp. $\mathrm{G} = \mathrm{U}(p, q)$) by an arithmetic subgroup Γ of standard type (see below). To simplify notation in the following discussion we will assume that the underlying number field is $\mathbf{Q}$ (resp. imaginary quadratic). The general case is treated in the body of the paper. Let V be the standard module for G and $\mathscr{S}(\mathrm{V}^n)$ denote the Schwartz space of the direct sum of n copies of V. In what follows $(\ ,\ )$ will denote a fixed symmetric (resp. hermitian) form on V of signature (p, q) such that G is the isometry group of $(\ ,\ )$. We also assume $p \geqslant q$ and $n \leqslant p$. Let G′ denote $\mathrm{Sp}_n(\mathbf{R})$ (resp. $\mathrm{U}(n, n)$) and $\widetilde{\mathrm{G}}'$ be the non-trivial 2-fold cover of G (resp. the non-trivial 2-fold cover which is trivial on $\mathrm{SU}(n, n)$). Then $\mathrm{G} \times \widetilde{\mathrm{G}}'$ operates on $\mathscr{S}(\mathrm{V}^n)$ by the (restriction of the) oscillator representation and consequently the continuous cohomology group $\mathrm{H}^{\bullet}_{ct}(\mathrm{G}, \mathscr{S}(\mathrm{V}^n))$ is a module for $\widetilde{\mathrm{G}}'$. Let $\mathfrak{k}'_0$ be the Lie algebra of the maximal compact subgroup $\widetilde{\mathrm{K}}'$ of $\widetilde{\mathrm{G}}'$ that covers $\mathrm{U}(n)$

(resp. $U(n) \times U(n)$) and $\mathfrak{p}'_0$ be its orthogonal complement in $\mathfrak{g}'_0$, the Lie algebra of G', computed using the Killing form. Then the projection $G' \to G'/K'$ identifies $\mathfrak{p}'_0$ with the tangent space to G'/K' at eK'. We let $\mathfrak{p}^-$ denote the subspace of $\mathfrak{p}'_0 \otimes \mathbf{C}$ which is identified with the anti-holomorphic tangent space to G'/K' at eK' (see § 6 for the definition of the complex structure on G'/K'). We let $\mathfrak{k}'$ be the complexification of $\mathfrak{k}'_0$ and $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}' = \mathfrak{g}'_0 \otimes \mathbf{C}$ given by $\mathfrak{q} = \mathfrak{k}' \oplus \mathfrak{p}^-$. We will use the following notation. If $\mathfrak{m}$ is a Lie algebra, $\chi : \mathfrak{m} \to \mathbf{C}$ is a homomorphism and U is an $\mathfrak{m}$-module, then $U^{\mathfrak{m}}_{\chi}$ will denote the subspace of U defined by

$$U^{\mathfrak{m}}_{\chi} = \{ u \in U : xu = \chi(x)\, u \ \text{ all } \ x \in \mathfrak{m} \}.$$

The next definition will play a very important role in this paper.

Definition. — A class $\varphi \in H^{\bullet}_{ct}(G, \mathscr{S}(V^n))$ which is annihilated by $\mathfrak{p}^-$ will be said to be holomorphic. The set of all such classes will be denoted $H^{\bullet}_{ct}(G, \mathscr{S}(V^n))^{\mathfrak{p}^-}$.

We next observe that there is a natural map

$$\theta : H^{\bullet}_{ct}(G, \mathscr{S}(V^n)) \to H^{\bullet}(\Gamma, \mathbf{C}).$$

Indeed we have the restriction map $H^{\bullet}_{ct}(G, \mathscr{S}(V^n)) \to H^{\bullet}(\Gamma, \mathscr{S}(V^n))$. But the Γ-invariant linear functional $\Theta : \mathscr{S}(V^n) \to \mathbf{C}$ (see § 3) induces a map $H^{\bullet}(\Gamma, \mathscr{S}(V^n)) \to H^{\bullet}(\Gamma, \mathbf{C})$. The map $\varphi \to \theta(\varphi)$ is then defined to be the composition of the two previous maps. Now let χ_m denote the character of $\widetilde{K}'$ given by $\chi_m(k') = \det(k')^{m/2}$ (resp. $\chi_m(k') = \det^+(k')^m \det^-(k')^{-m}$, where $\det^{\pm} : U(n) \times U(n) \to S^1$ are the determinants on the first and second factors). We may associate a $\widetilde{G}'$-homogeneous line bundle $\mathscr{L}_m$ over $\widetilde{G}'/\widetilde{K}'$ to the character χ_m above.

We now consider the bilinear pairing

$$((\ ,\ )) : H^i_c(M, \mathbf{C}) \times H^{a-i}_{ct}(G, \mathscr{S}(V^n)) \to C^{\infty}(\widetilde{G}')$$

given by

$$((\eta, \varphi))\,(g') = \int_M \eta \wedge \theta(\omega(g')\,\varphi).$$

Here we have set $a = \dim_{\mathbf{R}} M$ and we have represented $\theta(\omega(g')\,\varphi)$ by a closed $(a-i)$-form on M and η by a compactly-supported closed i-form on M. We will describe $((\ ,\ ))$ and θ more concretely in § 3.

It is well-known (§ 3) that if $\varphi \in H^{a-i}_{ct}(G, \mathscr{S}(V^n))^{\mathfrak{k}'}_{\chi_m}$, then we may identify $((\eta, \varphi))\,(g')$ with a ***harmonic*** (i.e. annihilated by $\square$ above) section of the quotient of $\mathscr{L}_m$ by a certain arithmetic lattice $\Gamma' \subset G'$—we will denote this quotient line bundle by $\mathscr{L}_m$ also. One of our main points is that if φ is a holomorphic class, then $((\eta, \varphi))\,(g')$ is identified with a ***holomorphic*** section of $\mathscr{L}_m$. We note that

$$H^{a-i}_{ct}(G, \mathscr{S}(V^n))^{\mathfrak{q}}_{\chi_m} = H^{a-i}_{ct}(G, \mathscr{S}(V^n))^{\mathfrak{p}^-} \cap H^{a-i}_{ct}(G, \mathscr{S}(V^n))^{\mathfrak{k}'}_{\chi_m}.$$

Before stating out first theorem we need to introduce a $\mathfrak{g}'$-invariant subspace of the continuous cohomology $H^{\bullet}_{ct}(G, \mathscr{S}(V^n))$. We recall [2, 5.1] that the choice of a maximal compact subgroup $K \subset G$ is equivalent to the choice of a positive definite form $(\ ,\ )_0$ on V which is a minimal majorant of the given form $(\ ,\ )$ on V of signature (p, q). We define the Gaussian $\varphi_0 \in \mathscr{S}(V^n)$ by

$$\varphi_0(v_1, \ldots, v_n) = \prod_{i=1}^{n} e^{-\pi(v_i, v_i)_0}.$$

Definition. — The polynomial Fock space $\mathbf{S}(V^n) \subset \mathscr{S}(V^n)$ is defined to be the space of those Schwartz functions on V^n of the form $p(v_1, \ldots, v_n)\, \varphi_0(v_1, \ldots, v_n)$, where $p(v_1, \ldots, v_n)$ is a polynomial function on V^n.

Remark. — The term "polynomial Fock space" is chosen because the image of $\mathbf{S}(V^n)$ under the intertwining map ι of § 6 from the Schrödinger model of the oscillator representation to the Fock model is the subspace $\mathscr{P}(\mathbf{C}^{nm})$ of holomorphic *polynomials* on $\mathbf{C}^{nm}$.

The subspace $\mathbf{S}(V^n)$ is invariant under the actions of $\mathfrak{g}$ and $\mathfrak{g}'$ (the complexified Lie algebras of G and G'). We recall, given a choice of K, that the van Est Theorem [4], IX, 5.6, gives a canonical isomorphism ν from continuous cohomology to relative Lie algebra cohomology

$$\nu : H^{\bullet}_{ct}(G, \mathscr{S}(V^n)) \to H^{\bullet}(\mathfrak{g}, K; \mathscr{S}(V^n)).$$

Definition. — We say that a class $\varphi \in H^{\bullet}_{ct}(G, \mathscr{S}(V^n))$ takes values in $\mathbf{S}(V^n)$ if $\nu(\varphi)$ is in the image of the natural map

$$H^{\bullet}(\mathfrak{g}, K; \mathbf{S}(V^n)) \to H^{\bullet}(\mathfrak{g}, K; \mathscr{S}(V^n)).$$

We may now state our first theorem. In what follows $m = p + q$.

Theorem **1**. — (i) *The induced pairing*

$$((\ ,\ )) : H^i_c(M, \mathbf{C}) \times H^{a-i}_{ct}(G, \mathscr{S}(V^n))^q_{\chi_m} \to \Gamma(\mathscr{L}_m)$$

takes values in the holomorphic sections; that is, in the space of Siegel (resp. hermitian) modular forms of weight $m/2$ (resp. m) and appropriate level.

(ii) *If $\eta \in H^i_c(M, \mathbf{C})$ and $\varphi \in H^{a-i}_{ct}(G, \mathscr{S}(V^n))^q_{\chi_m}$, then all Fourier coefficients a_β of $((\eta, \varphi))\,(g')$ are zero except the positive definite or positive semi-definite ones. Suppose further that φ takes values in $\mathbf{S}(V^n)$. Then there exist certain totally geodesic cycles C_β on M determined by β and independent of η and φ and associated to smaller orthogonal (resp. unitary) groups such that a_β is the period over C_β of the exterior product of η and a form $c_\beta(\varphi)$ determined by φ which comes from an invariant form on the universal cover of C_β.*

Remark. — This theorem is a somewhat formal consequence of our earlier papers [13] and [14] though a certain amount of work (§ 4) has to be done to compute the positive semi-definite Fourier coefficients. However it is by no means obvious that any holomorphic classes exist. Most of this paper is devoted to proving the existence of holomorphic classes

$\varphi^+_{nq} \in \mathrm{H}^{nq}_{ct}(\mathrm{O}(p, q), \mathscr{S}(\mathrm{V}^n))$ and $\varphi^+_{nq,nq} \in \mathrm{H}^{nq,nq}_{ct}(\mathrm{U}(p, q), \mathscr{S}(\mathrm{V}^n))$ which take values in $\mathbf{S}(\mathrm{V}^n)$. For these choices of φ the invariant forms $c_\beta(\varphi)$ above are the restrictions of invariant forms on D which depend only on n and q. Once one has made explicit the Fourier expansion of $((\eta, \varphi))$ for these choices of φ one obtains a large number of formulas analogous to those of Hirzebruch-Zagier [8] as we now describe.

The rest of this introduction is devoted to describing how the generating function for the intersection numbers of a fixed finite cycle C with the members of the family of locally finite cycles

$$\{ \mathrm{C}_\beta : \beta \in \mathrm{M}_n(\mathcal{O}),\ \beta^* = \beta,\ \beta \geqslant 0 \text{ at all infinite places} \}$$

described in § 2 is a Siegel (or hermitian) modular form. Our method of proof is to use the previous correspondence with $\varphi = \varphi^+_{nq}$ (resp. $\varphi^+_{nq,nq}$) and η the Poincaré dual of C to construct a holomorphic modular form, which is then shown to have the above generating function as its Fourier series. Our results are the analogues of those of Hirzebruch-Zagier [8] for the Hilbert modular subgroups of O(2, 2) and Shintani [21] for O(2, 1).

Let $\mathbf{k}$ be a totally real number field of degree r, let $\mathcal{O}$ be the ring of integers in $\mathbf{k}$, and let V be a vector space of dimension m over $\mathbf{k}$. Let R be the set of archimedean embeddings of $\mathbf{k}$. Let (,) be a quadratic form on V with signature (p, q) at one archimedean embedding of V and positive definite at the others. Let β be a symmetric $n \times n$ matrix with entries in $\mathcal{O}$ that is positive semi-definite at all archimedean embeddings of $\mathbf{k}$. Then as described in § 2 we can construct a locally-finite cycle C_β in the arithmetic quotient $\mathrm{M} = \Gamma\backslash\mathrm{D}$ of the symmetric space D of $\mathrm{O}(p, q)$. Here Γ is a congruence subgroup of the group of units of (,). The cycle C_β is of dimension $(p - t)\,q$ where $t = \operatorname{rank} \beta$. If q is even there exists an invariant q-form e_q, the Euler form, on D (see § 9 for the definition). We define $e_q = 0$ if q is odd. Suppose η is a closed rapidly decreasing $(p - n)\,q$-form on M (if M is compact all η are considered to be rapidly decreasing). See [3] for the meaning of the term "rapidly-decreasing". If $\operatorname{rank} \beta = t$, then $\dim \mathrm{C}_\beta = (p - t)\,q$ and $\deg \eta \wedge e_q^{n-t} = (p - t)\,q$ and we can take the period $\int_{\mathrm{C}_\beta} \eta \wedge e_q^{n-t}$. We define a generating function $\mathrm{P}(\tau, \eta)$ for these periods with $\tau \in \mathfrak{h}_n^r$ (here $\mathfrak{h}_n$ denotes the Siegel upper half-space of genus n). We let $\mathscr{L}$ denote the lattice of symmetric $n \times n$ matrices with entries in $\mathcal{O}$ in the product of r copies of the real $n \times n$ symmetric matrices. We let $\mathscr{L}(t)$ denote the set of those elements of $\mathscr{L}$ which have rank t and are positive semi-definite at all places of $\mathbf{k}$. We will employ the following notation throughout. If $z \in \mathbf{C}$, we will abbreviate $\exp(2\pi i z)$ to $e(z)$. If $z = (z_\lambda) \in \mathrm{M}_n(\mathbf{C})^r$, then $e_*(\beta z)$ will denote $e(1/2 \operatorname{tr} \Sigma_{\lambda \in \mathrm{R}} \lambda(\beta)\, z_\lambda)$. We can now define $\mathrm{P}(\tau, \eta)$

$$\mathrm{P}(\tau, \eta) = \sum_{t=0}^{n} \sum_{\beta \in \mathscr{L}(t)} \left(\int_{\mathrm{C}_\beta} \eta \wedge e_q^{n-t} \right) e_*\,(\beta\tau).$$

Remark. — If q is odd the sum is only over β of rank n.

We then have the following theorem, generalizing Shintani [21].

Theorem **2.** — *The function* $P(\tau, \eta)$ *is a holomorphic modular form of weight* $m/2$ *for a suitable congruence subgroup of* $\mathrm{Sp}_n(\mathcal{O})$. *If* q *is odd,* $P(\tau, \eta)$ *is a cusp form.*

There is also a homology version of Theorem 2. Let C be a finite (compact) nq-cycle in M. Then the Poincaré dual cohomology class to C has a compactly supported representative. Substituting this form for η in the above theorem, we obtain the following analog of Hirzebruch-Zagier [8]. Define $I(\tau, C)$ for $\tau \in \mathfrak{h}_n^r$ by

$$I(\tau, C) = \sum_{t=0}^{n} \sum_{\beta \in \mathscr{L}(t)} C.(C_\beta \cap e_q^{n-t})\, e_*(\beta\tau).$$

Here . denotes the intersection product of cycles and $\cap$ is the (right) cap-product between cohomology and homology; that is, we have for $\deg C = (p-t)\,q$, $\deg \eta = (p-n)\,q$ and $\deg \omega = (n-t)\,q$

$$\int_C \eta \wedge \omega = \int_{C \cap \omega} \eta.$$

Theorem **2** *(bis)*. — *The function* $I(\tau, C)$ *is a holomorphic modular form of weight* $m/2$ *for a suitable congruence subgroup of* $\mathrm{Sp}_n(\mathcal{O})$. *If* q *is odd,* $I(\tau, C)$ *is a cusp form.*

We have a corresponding theorem for the symmetric spaces of the unitary groups $U(p, q)$. Let F be a totally imaginary quadratic extension of the field $\mathbf{k}$ above, let $\mathcal{O}$ be the ring of integers in F, and let V be a vector space of dimension $m = p + q$ over F. Let R be a cross-section for the action of complex conjugation on the archimedean embeddings of F. Let (,) be a hermitian form on V with signature (p, q) at one complex conjugacy class of archimedean embeddings of F and positive definite at the others. Let β be a hermitian $n \times n$ matrix over $\mathcal{O}$ that is positive semi-definite at all archimedean embeddings of F. Again we have locally-finite cycles C_β for each such matrix β in arithmetic quotients $M = \Gamma\backslash D$ of the symmetric space D of $U(p, q)$. Here Γ is a congruence subgroup of the group of units of (,). The cycle C_β is of complex dimension $(p-t)\,q$ where $t = \operatorname{rank} \beta$. For all q there exists an invariant $2q$-form c_q, the qth Chern form, on D (see § 9 for the definition of c_q).

Now suppose η is a closed rapidly decreasing $2(p-n)\,q$ form on M. If M is compact, all η are considered to be of rapid decrease. If $\operatorname{rank} \beta = t$ we can take the period $\int_{C_\beta} \eta \wedge c_q^{n-t}$. We define the generating function $P(\tau, \eta)$ for these periods as β runs through the lattice $\mathscr{L}$ of hermitian $n \times n$ matrices with entries in $\mathcal{O}$ in the product of r copies of the hermitian $n \times n$ matrices. We let $\mathscr{L}(t)$ denote the set consisting of those elements of $\mathscr{L}$ that have rank t and are positive semi-definite at all places of $\mathbf{k}$. We will use the symbol $U(n, n)$ to denote the isometry group of the standard split skew-hermitian form $\langle\ ,\ \rangle$ on $\mathbf{C}^{2n}$. We have, for $e_1, e_2, \ldots, e_{2n}$ the standard basis,

$$\langle \sum_{j=1}^{2n} z_j e_j, \sum_{j=1}^{2n} w_j e_j \rangle = \sum_{j=1}^{n} (z_j \overline{w}_{n+j} - z_{n+j} \overline{w}_j).$$

We let $\mathbf{C}\mathfrak{h}_n$ be the symmetric space of $\mathrm{U}(n, n)$, the space of complex $n \times n$ matrices $\tau = u + iv$ with positive definite skew-hermitian part (i.e., u is hermitian, iv is skew-hermitian, and v is positive definite). Let $\tau \in (\mathbf{C}\mathfrak{h}_n)^r$. We define

$$\mathrm{P}(\tau, \eta) = \sum_{t=0}^{n} \sum_{\beta \in \mathscr{L}(t)} \left(\int_{\mathrm{C}_\beta} \eta \wedge c_q^{n-t} \right) e_*(\beta\tau).$$

We have the following theorem.

Theorem **3.** — *The function* $\mathrm{P}(\tau, \eta)$ *is a holomorphic modular form of weight* m *for a suitable congruence subgroup of the group of* $\mathcal{O}$*-points of* $\mathrm{U}(n, n)$.

We have a corresponding theorem in terms of intersection numbers of cycles. Let C be a finite (compact) $2nq$-cycle. Define $\mathrm{I}(\tau, \mathrm{C})$ for $\tau \in (\mathbf{C}\mathfrak{h}_n)^r$ by

$$\mathrm{I}(\tau) = \sum_{t=0}^{n} \sum_{\beta \in \mathscr{L}(t)} \mathrm{C}.(\mathrm{C}_\beta \cap c_q^{n-t})\, e_*(\beta\tau).$$

Theorem **3** *(bis).* — *The function* $\mathrm{I}(\tau, \mathrm{C})$ *is a holomorphic modular form of weight* m *for a suitable congruence subgroup of the group of* $\mathcal{O}$*-points of* $\mathrm{U}(n, n)$.

In order to make a true generalization of the results of Hirzebruch-Zagier [8] one is forced to give up the hypothesis that η has rapid decrease. This creates enormous complications. First the integral defining $\theta_\varphi(\eta)$ (see § 3 for the meaning of this symbol) usually does not converge. Even if the integral converges, or one defines the integral by regularization, [18], the automorphic form $\theta_\varphi(\eta)$ will no longer necessarily be holomorphic; there will be a formula for $\bar\partial\theta_\varphi(\eta)$ involving an integral over the " boundary " of $\Gamma\backslash\mathrm{D}$ (recall that $\bar\partial$ applied to the integrand defining $\theta_\varphi(\eta)$ was not zero but was exact). Finally, cycles associated to the isotropic vectors will contribute to the singular Fourier coefficients. It seems likely that Eisenstein cohomology will make an appearance at this point. Cogdell [5] has analyzed the situation in the case M is a finite-volume quotient of the 2-ball in terms of the desingularization of the Baily-Borel compactification. The second author [18] has shown that for the case $\mathrm{G} = \mathrm{O}(p, 1)$ the singular Fourier coefficients of $((\eta, \varphi))$ involve periods of η over certain tori in the Borel-Serre boundary. It will be interesting to see if there is a compactification that will allow a similar analysis in the general case.

We would like to thank Mark Stern for suggesting the proof of Lemma 4.2, Roger Howe for suggesting that we look at the " Howe operators " of § 5 and G. Harder, F. Hirzebruch, J. Schwermer and D. Zagier for encouragement and hospitality at the Max-Planck Institut für Mathematik in Bonn where a considerable amount of the work on this paper was done. Also we should mention the work of Y. L. Tong and S. P. Wang [22-26] running parallel to our papers [9-14]. Finally, we would like to thank the referee for an extraordinarily thorough critique of our original manuscript.

1. Special cycles

In what follows $D = G/K$ will be a Riemannian symmetric space with G a semi-simple Lie group having no compact factors and $K \subset G$ a maximal compact subgroup. We let $\Gamma \subset G$ be a torsion-free lattice and $M = \Gamma \backslash D$ be the associated symmetric space. Let $\pi : D \to M$ be the covering projection. Now suppose σ_1 is an isometry of D of order 2 such that

$$\sigma_1 \Gamma \sigma_1 = \Gamma.$$

Let D_1 be the fixed point set of σ_1 in D, and Let Γ_1, G_1, K_1 be the fixed-point sets of σ_1 in Γ, G, and K respectively. Then D_1 is a totally geodesic subsymmetric space of D isomorphic to G_1/K_1. We let $M_1 = \Gamma_1 \backslash D_1$ be the corresponding locally symmetric space. We assume that Γ has been chosen so that M_1 is orientable (this can present a problem, but we have been able to deal with it in the cases of interest here). The restriction of π to D_1 induces a map $j_1 : M_1 \to M$.

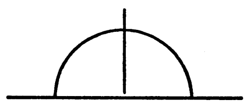
Lemma **1.1.** — *Suppose* Γ *is arithmetic and* G_1 *is defined over* $\mathbf{Q}$. *Then* j_1 *is a proper embedding onto a totally geodesic submanifold of* M.

Proof. — The proof of Lemma 2.7 of [1] is valid for any rational reductive $M \subset G$ (notation of [1]). But G_1 above is rational and reductive. ∎

We call the locally finite cycle represented by the image of j_1 a *special cycle*. If M is compact, then M_1 will also be compact. The image of j_1 is a component of the fixed-point set of σ_1 acting on M.

Millson and Raghunathan were led to consider such cycles because of remarkable property they satisfy. *They come in complementary pairs.* In fact this is true only under the further condition (usually satisfied in practice) that there exist " rational points " on D_1. By this we mean a point $x \in D_1$ such that the associated Cartan involution θ satisfies $\theta\Gamma\theta \subset \Gamma$. In this case we put $\sigma_2 = \theta\sigma_1$ and define D_2 to be the fixed-point set of σ_2. If M is compact, the image M_2 of D_2 in M is compact. If the intersection number of M_1 and M_2 is nonzero, neither is a boundary.

Example.



Here D is the upper half-plane and σ_1 and σ_2 are reflections in the y-axis and unit circle respectively.

Using the above idea, Millson and Raghunathan were able to give many examples of nonvanishing cohomology groups for locally symmetric spaces [19].

2. Special cycles in orthogonal and unitary locally symmetric spaces

In this paper we will be concerned only with the cases in which D is the symmetric space of $O(p, q)$ or $U(p, q)$. In this case we can take advantage of the projective structure of D to better understand special cycles. Let $\mathbf{k}$ be a totally real field (respectively, totally imaginary quadratic extension of a totally real field) and $\mathcal{O}$ be the ring of integers in $\mathbf{k}$. Let $\mathcal{R} = \{\lambda_1, \ldots, \lambda_r\}$ be the set of archimedean embeddings (resp. archimedean embeddings up to conjugation) of $\mathbf{k}$. In the second case, let $\mathbf{k}_0$ be the totally real subfield of $\mathbf{k}$ fixed by complex conjugation. Let $V_{\mathbf{k}}$ be a vector space over $\mathbf{k}$ and $L \subset V_{\mathbf{k}}$ an $\mathcal{O}$-lattice. Let $(\ , \ )$ be a nondegenerate quadratic (respectively, hermitian) form on $V_{\mathbf{k}}$, which is integral ($\mathcal{O}$-valued) on L and has signature (p, q) at one archimedean completion of $\mathbf{k}$ and is positive definite at all other classes of archimedean completions of $\mathbf{k}$. We let V be the vector space of real points of the vector space obtained from $V_{\mathbf{k}}$ by restriction of scalars from $\mathbf{k}$ to $\mathbf{Q}$ (resp. $\mathbf{k}_0$ to $\mathbf{Q}$). Then we have an isomorphism of real vector spaces

$$V \simeq \bigoplus_{\alpha=1}^{r} V^{(\alpha)}$$

where $V^{(\alpha)}$ is the completion of $V_{\mathbf{k}}$ corresponding to λ_α. We will assume that the form $(\ , \ )_1$ induced on $V^{(1)}$ by $(\ , \ )$ has signature (p, q), and consequently that the form $(\ , \ )_\alpha$ on $V^{(\alpha)}$ induced by $(\ , \ )$ is positive definite for $\alpha \geqslant 2$. We will abuse notation and use $(\ , \ )$ to denote the induced form on V. This form is the orthogonal direct sum of the forms $(\ , \ )_\alpha$ for $\alpha = 1, 2, \ldots, r$. We will again abuse notation and let L denote the $\mathbf{Z}$-lattice in V induced by the $\mathcal{O}$-lattice L in $V_{\mathbf{k}}$. Finally, we let $G^{(\alpha)}$ denote the subgroup of $\mathrm{Aut}(V^{(\alpha)})$ consisting of isometries of $(\ , \ )_\alpha$, and let $G = \prod_{\alpha=1}^{r} G^{(\alpha)}$. Then G is isomorphic to the real points of the reductive group over $\mathbf{Q}$ obtained from the isometries of the form $(\ , \ )$ on $V_{\mathbf{k}}$ by restriction of scalars from $\mathbf{k}$ to $\mathbf{Q}$ (resp. $\mathbf{k}_0$ to $\mathbf{Q}$). We note that $G^{(1)} \simeq O(p, q)$ (resp. $U(p, q)$) and $G^{(\alpha)} \simeq O(m)$ (resp. $U(m)$) for $\alpha \geqslant 2$.

Let Γ denote a torsion-free congruence subgroup of the subgroup of $GL(V)$ preserving L and $(\ , \ )$. Let $U_{\mathbf{k}} \subset V_{\mathbf{k}}$ be an oriented rational subspace such that $(\ , \ ) \mid U_{\mathbf{k}}$ is nondegenerate, and let $U \subset V$ be the space of real points of the $\mathbf{Q}$-vector space obtained from U by restriction of scalars. Then we will construct special cycles $C_U \subset \Gamma \backslash D$, where D is the symmetric space of the isometry group G of $(\ , \ )$. Since $(\ , \ ) \mid U$ is nondegenerate, we have a direct sum decomposition $V = U + U^{\perp}$. Recall that we may consider D as the open subset of the Grassmannian $\mathrm{Gr}_q(V)$ consisting of those q-planes Z such that $(\ , \ ) \mid Z$ is negative definite. We may now define a subset $D_U \subset D$ by

$$D_U = \{ Z \in D : Z = Z \cap U + Z \cap U^{\perp} \}$$

We let G_U denote the stabilizer of U in G and G_U^0 be the connected component of the identity in G_U. We put $\Gamma_U = \Gamma \cap G_U$ and $\Gamma_U^0 = \Gamma \cap G_U^0$. We let $C_U = \Gamma_U^0 \backslash D_U$ whence C_U is an orientable manifold. Since $\pi : C_U \to \Gamma \backslash D = M$ is proper, the pair (C_U, π) is a (locally—finite) singular cycle. We have the following elementary lemma which " resolves the singularities of $\pi(C_U)$ ".

Lemma **2.1.** — *There exists a normal subgroup of finite index* $\Gamma' \subset \Gamma$ *such that the upper horizontal arrow in the following diagram is an embedding.*

$$\begin{array}{ccc} \Gamma'_U\backslash D_U & \xrightarrow{\pi} & \Gamma'\backslash D \\ \downarrow & & \downarrow \\ \Gamma_U\backslash D_U & \xrightarrow{\pi} & \Gamma\backslash D \end{array}$$

Proof. — Let σ_U be the involution which is $+1$ on U and -1 on $U^\perp$ and let Γ' be a normal subgroup of finite index in Γ which is normalized by σ_U. The lemma follows from the "Jaffee Lemma", Proposition 2.2 of [19]. ∎

Remarks. — It is still possible in the case $G^{(1)} = SO(p, q)$ that $\Gamma'_U\backslash D_U$ is not orientable. However we may deal with this problem in the case that will concern us here as follows. Suppose $(\, , \,) \mid U$ is positive define. We may choose Γ' so that the rational spinor norm of every element of Γ' is 1 [19], Proposition 4.1. In this case $\Gamma'_U\backslash D_U$ is orientable. In summary, at least for the case $(\, , \,) \mid U$ positive definite, up to finite coverings the singular cycle (C_U, π) is carried by an embedded oriented submanifold of $M = \Gamma\backslash D$. We will abuse notation and use the symbol C_U instead of (C_U, π).

We now explain how an orientation of U gives rise to an orientation of D_U. The following discussion of orientability involves only $V^{(1)}$. To simplify notation, we will assume for the next five paragraphs that $r = 1$. We will need a rule for orienting the tensor product $V \otimes W$ of two oriented vector spaces V and W. We will use the rule that if $\{ v_1, \ldots, v_m \}$ is a properly oriented basis for V and $\{ w_1, \ldots, w_n \}$ is a properly oriented basis of W, then the basis $\{ v_i \otimes w_j \mid 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n \}$ ordered by the lexicographic order read from right to left (so $v_2 \otimes w_1$ comes before $v_1 \otimes w_2$) is a properly oriented basis for $V \otimes W$. Since V^* is canonically oriented if V is (the dual of a properly oriented basis is defined to be properly oriented) we obtain an orientation on $\mathrm{Hom}(V, W) \simeq V^* \otimes W$. We choose a basepoint $Z_0 \in D$ and choose an orientation of Z_0 and an orientation of V once and for all. Propagate the orientation of Z_0 continuously to orient all other $Z \in D$. Orienting D is equivalent to giving an orientation of $\mathrm{Hom}(Z, Z^\perp)$ which depends continuously on Z. Since V is oriented we obtain an induced orientation of $Z^\perp$ such that the orientation of $Z^\perp$ followed by that of Z is the orientation of V. Then $T_Z(D) \simeq \mathrm{Hom}(Z, Z^\perp)$ is oriented. With our conventions, if we choose the basis

$$\{ \omega_{\alpha\mu} \mid 1 \leqslant \alpha \leqslant p, p + 1 \leqslant \mu \leqslant p + q \}$$

of § 5 for $T^*_{Z_0}(D) \simeq \mathfrak{p}^*$, then the element

$$\omega_{1, p+1} \wedge \ldots \wedge \omega_{1, p+q} \wedge \ldots \wedge \omega_{p, p+1} \wedge \ldots \wedge \omega_{p, p+q}$$

is a properly oriented basis element for $\Lambda^{pq} T^*_{Z_0}(D)$ in the orthogonal case.

Now it is easily seen that there are canonical isomorphisms of the tangent space $T_Z(D_U)$ and the normal space $\nu_Z(D_U)$

$$T_Z(D_U) \cong \mathrm{Hom}(Z \cap U, Z^\perp \cap U) + \mathrm{Hom}(Z \cap U^\perp, Z^\perp \cap U^\perp),$$

and

$$\nu_Z(D_U) \cong \mathrm{Hom}(Z \cap U, Z^\perp \cap U^\perp) + \mathrm{Hom}(Z \cap U^\perp, Z^\perp \cap U).$$

We will now use the above considerations and orientation of U to orient $\nu_Z(D_U)$. Then $T_Z(D_U)$ will receive an orientation by the rule that the orientation of $T_Z(D_U)$ followed by the orientation of $\nu_Z(D_U)$ is the orientation of $T_Z(D)$. Since the problem of orienting $\nu_Z(D_U)$ is somewhat complicated for general U, we will discuss only the case in which $(\ ,\ ) \mid U$ is positive definite. In this case we have

$$T_Z(D_U) \simeq \mathrm{Hom}(Z, Z^\perp \cap U^\perp)$$

and

$$\nu_Z(D_U) \simeq \mathrm{Hom}(Z, U).$$

Thus we may orient $\nu_Z(D_U)$ by the rule indicated above.

The signature of $(\ ,\ ) \mid U$ plays an important role in understanding the nature of the class C_U. Suppose then that the signature of $(\ ,\ ) \mid U$ is (r, s); whence $\dim U = r + s$. The case in which r or s is zero is of special importance. In case $s = 0$, then U is a positive r-plane (so $r \leqslant p$) and

$$D_U = \{ Z \in D : Z \subset U^\perp \}.$$

In case $r = 0$, then U is a negative s-plane (so $s \leqslant q$) and

$$D_U = \{ Z \in D : Z \supset U \}.$$

We say such special cycles are of *definite type*. We say the other cycles are of *mixed type*.

Example. — $D = \mathbf{H} \times \mathbf{H}$. In this case the definite cycles correspond to quotients of linearly embedded upper half-planes. The mixed cycles correspond to products of two geodesics; they are totally real geodesic tori.

In what follows we will use certain linear combinations $C_\varnothing$ of the cycles C_U. In order to define these we introduce some terminology. If $X = (x_1, x_2, \ldots, x_n) \in V^n$, then (X, X) will denote the symmetric (resp. hermitian) matrix such that

$$(X, X)_{ij} = (x_i, x_j).$$

If $r > 1$ we will need to keep track of the index in the decomposition

$$V = \bigoplus_{\alpha=1}^{r} V^{(\alpha)},$$

and the corresponding decomposition

$$X = (X^{(1)}, X^{(2)}, \ldots, X^{(r)}).$$

The symbol (X, X) will mean the r-tuple of $n \times n$ symmetric (resp. hermitian) matrices given by

$$(X, X) = ((X^{(1)}, X^{(1)}), \ldots, (X^{(r)}, X^{(r)})).$$

Thus if $\beta = (\beta^{(1)}, \ldots, \beta^{(r)})$ is an r-tuple of such matrices, the equation $(X, X) = \beta$ means termwise equality.

We now wish to discuss the G-orbits of V^n.

Definition. — The orbit $\mathcal{O} = GX$ is

(i) *non-singular* if rank $(X^{(\alpha)}, X^{(\alpha)}) = n$ for $\alpha = 1, 2, \ldots, r$,

(ii) *non-degenerate* if rank $(X^{(\alpha)}, X^{(\alpha)}) = \dim_{\mathbf{R}} \operatorname{span} X^{(\alpha)}$ (resp. $\dim_{\mathbf{C}} \operatorname{span} X^{(\alpha)}$), that is $(\ , \ ) \mid \operatorname{span} X^{(\alpha)}$ is non-degenerate, for $\alpha = 1, 2, \ldots, r$,

(iii) *degenerate* if $(\ , \ ) \mid \operatorname{span} X^{(\alpha)}$ is degenerate for some $\alpha \in \{1, 2, \ldots, r\}$.

We note that the zero orbit is non-degenerate but singular. We note also that an orbit $\mathcal{O}$ is closed if and only if it is non-degenerate.

Now suppose $\mathcal{O} \subset V^n$ is closed G-orbit. Then by Borel [2], Theorem 9.11, $\mathcal{O} \cap L^n$ consists of a finite number of Γ-orbits. We choose Γ-orbit representatives $\{Y_1, Y_2, \ldots, Y_\ell\}$ and let $U_j = \operatorname{span} Y_j$. We then define $C_{\mathcal{O}}$ by

$$C_{\mathcal{O}} = \sum_{j=1}^{n} C_{U_j}.$$

We have explained how an orientation of U_j induces an orientation of C_{U_j}. In case the elements of Y_j are independent we give U_j the orientation determined by Y_j. Otherwise we refine Y_j to a basis $\mathscr{B}_j$ starting at the left. More precisely, $\mathscr{B}_j$ is defined as follows. Suppose $Y_j = \{Y_{j1}, \ldots, Y_{jn}\}$. Then $Y_{j1} \in \mathscr{B}_j$ if and only if $Y_{j1} \neq 0$. Also $Y_{ji} \in \mathscr{B}_j$ if and only if $Y_{ji} \notin \operatorname{span}\{Y_{j1}, \ldots, Y_{j(i-1)}\}$.

We now define $\mathscr{Q}_\beta \subset V^n$ for $\beta = (\beta^{(1)}, \ldots, \beta^{(r)})$ an r-tuple of n by n symmetric (resp. hermitian) matrices by

$$\mathscr{Q}_\beta = \{X \in V^n : (X, X) = \beta\}.$$

Clearly if $X \in \mathscr{Q}_\beta$, then $\mathcal{O} = GX \subset \mathscr{Q}_\beta$. In case $\beta^{(\alpha)}$ is positive definite for all α, then G acts transitively on $\mathscr{Q}_\beta$ and $\mathcal{O} = \mathscr{Q}_\beta$. In case β is a totally positive definite symmetric (resp. hermitian) matrix over $\mathbf{k}$ we will henceforth identify β with the r-tuple $(\lambda_1(\beta), \ldots, \lambda_r(\beta))$ and we will write C_β instead of $C_{\mathcal{O}}$ for the cycle corresponding to $\mathcal{O} = \mathscr{Q}_\beta$. In this case C_β is a locally finite cycle such that each irreductible component has real dimension $(p - n)\, q$ (resp. $2(p - n)\, q$).

Suppose now that β is positive semidefinite and has rank t with $t < n$. Then $\mathscr{Q}_\beta$ is a union of G-orbits. However it contains a unique closed orbit $\mathscr{Q}_\beta^c$ described as follows:

$$\mathscr{Q}_\beta^c = \{X \in \mathscr{Q}_\beta : \dim \operatorname{span} X^{(\alpha)} = t, \text{ for all } \alpha\}.$$

Lemma **2.2.** — *The group* G *operates transitively on* $\mathscr{Q}_\beta^c$.

Proof. — It suffices to consider the case $r = 1$ by successively applying the argument for $r = 1$ to the components $X^{(\alpha)}$ and $(X')^{(\alpha)}$.

Let $X, X' \in \mathscr{Q}_\beta^c$. Let $U = \operatorname{span} X$, $U' = \operatorname{span} X'$. We claim that the map $f : X \to X'$ given by $f(x_j) = x'_j$, $1 \leqslant j \leqslant n$, extends to a linear map $F : U \to U'$. Suppose that

$c_1 x_1 + \ldots + c_n x_n = 0$ is a dependence relation among $x_1, \ldots, x_n$. Let c be the row-vector $(c_1, c_2, \ldots, c_n)$. Then the row-vector $c\beta = 0$ because it represents the linear functional $(c_1 x_1 + \ldots + c_n x_n, .)$ on U relative to the spanning set $x_1, \ldots, x_n$. But then the linear functional $(c_1 x'_1 + \ldots + c_n x'_n, .)$ on U′ is also zero because it is also represented by the row-vector $c\beta$. Since $(\ ,\ ) \mid \mathrm{U}'$ is nondegenerate we conclude that $c_1 x'_1 + \ldots + c_n x'_n = 0$. ■

If β is a totally positive semi-definite matrix with entries in $\mathbf{k}$, we will use the above identification and let C_β denote the cycle $\mathrm{C}_{\mathcal{O}}$ associated to the closed orbit $\mathcal{Q}^c_\beta$.

As it stands C_β would frequently be zero for trivial reasons. Indeed pairs of frames $\mathrm{Y} = (x_1, \ldots, x_n)$ and $\mathrm{Y}' = (-x_1, x_2, \ldots, x_n)$ would occur in $\mathcal{Q}_\beta$ if β were diagonal. To avoid such cases where C_β would be trivially zero we modify its definition by introducing a congruence condition. Let $h \in \mathrm{L}^n$ and $\mathfrak{a} \subset \mathcal{O}$ be an ideal. Then we replace $\mathcal{Q}_\beta \cap \mathrm{L}^n$ above by $\mathcal{Q}_\beta \cap (h + \mathfrak{a}\mathrm{L}^n)$. We assume that $\gamma \in \Gamma$ implies $\gamma \equiv 1 \bmod \mathfrak{a}$ so that Γ operates on this intersection. Once again there are a finite number of Γ-orbits. We choose Γ-orbit representatives $\mathscr{C}_{\beta, h} = \{\mathrm{Y}_1, \mathrm{Y}_2, \ldots, \mathrm{Y}_{\ell'}\}$ and proceed as above to get a cycle

$$\mathrm{C}_{\beta, h} = \sum_{j=1}^{\ell'} \mathrm{C}_{\mathrm{U}_j}.$$

In case $\mathrm{G} = \mathrm{U}(p, q)$, the cycle $\mathrm{C}_{\beta, h}$ is an algebraic cycle.

We now choose $h \in \mathrm{L}^n$ and $\mathfrak{a} \subset \mathcal{O}$ as above once and for all and obtain cycles as above. Since h and $\mathfrak{a}$ are fixed we drop the h in $\mathrm{C}_{\beta, h}$; henceforth C_β means the cycle constructed as above using elements $\mathrm{X} \in h + \mathfrak{a}\mathrm{L}^n$.

3. The Cauchy-Riemann equations and the theta correspondence

In this section we discuss a cohomological version of the theta correspondence. We will be particularly interested in proving that integrals of cohomological type depending on a symplectic (or split unitary) parameter are holomorphic. It is convenient to take the viewpoint of continuous cohomology in this section. A basic reference for continuous cohomology is Borel-Wallach [4].

Let G be as in the beginning of the previous section and G′ be the group of real points of the restriction of scalars from $\mathbf{k}$ to $\mathbf{Q}$ (resp. from $\mathbf{k}_0$ to $\mathbf{Q}$) of $\mathrm{Sp}(n, \mathbf{k})$ (resp. $\mathrm{U}(n, n)$ $(\mathbf{k}_0)$). Let $\widetilde{\mathrm{G}}'$ be the twofold cover of G′ whose restriction to each simple factor is the 2-fold cover described in the introduction.

We consider the continuous cohomology groups $\mathrm{H}^\bullet_{ct}(\mathrm{G}, \mathscr{S}(\mathrm{V}^n))$. Here G operates on $\mathscr{S}(\mathrm{V}^n)$, the space of complex-valued Schwartz functions on V^n, by the action ρ given by

$$\rho(g)\, \varphi(x) = \alpha(g)\, \varphi(g^{-1} x).$$

Here α is a character of G which is trivial in case $\mathrm{G}^{(1)} = \mathrm{U}(p, q)$ and, in case $\mathrm{G}^{(1)} = \mathrm{O}(p, q)$, is the character $\varepsilon^n \otimes 1 \otimes \ldots \otimes 1$ where ε is the spinor norm. Recall that by the van

Est theorem [4], the above cohomology groups may be realized as the de Rham cohomology of the complex $C^\bullet$, d whose ith cochain group is given by

$$C^i = (\mathscr{A}^i(D) \otimes \mathscr{S}(V^n))^G.$$

Here the superscript G denotes the G-invariants and $\mathscr{A}^i(D)$ denotes the smooth differential i-forms in D. The differential d is defined on a decomposable $\nu \otimes \varphi$ by

$$d(\nu \otimes \varphi) = (d\nu) \otimes \varphi,$$

where the d on the right-hand side is the usual exterior differential. We will identify a continuous cohomology class with a class of closed differential forms on D with values in $\mathscr{S}(V^n)$. If φ is such a differential form, we let $[\varphi]$ denote its cohomology class. We observe that we have an isomorphism given by evaluation at a point $Z_0 \in D$:

$$(\mathscr{A}^i(D) \otimes \mathscr{S}(V^n))^G \cong (\Lambda^i \mathfrak{p}^* \otimes \mathscr{S}(V^n))^K.$$

If $\varphi \in (\Lambda^i \mathfrak{p}^* \otimes \mathscr{S}(V^n))^K$ we will use $\varphi(Z, X)$ with $Z \in D$, $X \in V^n$ to denote the corresponding element in $(\mathscr{A}^i(D) \otimes \mathscr{S}(V^n))^G$. Here K is the maximal compact subgroup of G that is the isotropy subgroup of Z_0 and $\mathfrak{p}$ is the orthogonal complement of $\mathfrak{k}$, the complexified Lie algebra of K, in $\mathfrak{g}$, the complexified Lie algebra of G, for the Killing form on $\mathfrak{g}$.

The group $\widetilde{G}'$ operates on $\mathscr{S}(V^n)$ by the oscillator (or Weil) representation ω, see [15] or [27]. This action commutes with the action ρ of G, and hence $\widetilde{G}'$ operates on $H^i_{ct}(G, \mathscr{S}(V^n))$. We let $\widetilde{K}'$ denote the maximal compact subgroup of $\widetilde{G}'$ lying over $U(n)^r \subset Sp_n(\mathbf{R})^r$ (respectively $(U(n) \times U(n))^r \subset U(n, n)^r$). We let $\mathfrak{p}'$ denote the orthogonal complement to $\mathfrak{k}'$, the complexification of the Lie algebra of $\widetilde{K}'$, in $\mathfrak{g}'$ the complexified Lie algebra of G′ for the Killing form of $\mathfrak{g}'$. Let D′ be the symmetric space of $\widetilde{G}'$. Then D′ is Hermitian symmetric. Since we may identify $\mathfrak{p}'$ with the complexified tangent space to D′ at the identity coset, we have a splitting of $\mathfrak{p}'$ into the holomorphic and antiholomorphic tangent spaces

$$\mathfrak{p}' = \mathfrak{p}^+ + \mathfrak{p}^-.$$

We observe that $\mathfrak{p}^-$ acts on $H^\bullet_{ct}(G, \mathscr{S}(V^n))$ by the action of ω on the coefficients.

Definition. — We will say that the cohomology class $[\varphi] \in H^\bullet_{ct}(G, \mathscr{S}(V^n))$ is holomorphic if it is annihilated by $\mathfrak{p}^-$ under the action described above.

We will also want so specify the transformation law under $\widetilde{K}'$. To this end we let χ_m be the r-fold external tensor power of the character of $\widetilde{K}'$ described in the introduction and $\mathbf{C}_{\chi_m}$ be the 1-dimensional module on which $\widetilde{K}'$ acts according to the character χ_m. We then consider the double complex $C^{i,j}$, d, $\bar\partial$ given by

$$C^{i,j} = (\Lambda^i \mathfrak{p}^* \otimes \Lambda^j(\mathfrak{p}^-)^* \otimes \mathscr{S}(V^n) \otimes \mathbf{C}_{\chi_m})^{K \times \widetilde{K}'}.$$

In order to describe the operators d and $\bar\partial$ we introduce the following notation. Let $\{ X_i \mid 1 \leqslant i \leqslant N \}$ be a basis for $\mathfrak{p}$ and $\{ \omega_i \mid 1 \leqslant i \leqslant N \}$ be the dual basis.

Let $\{\bar{Z}_j \mid 1 \leqslant j \leqslant N'\}$ be a basis for $\mathfrak{p}^-$ and $\{\eta_j \mid 1 \leqslant j \leqslant N'\}$ be the dual basis. Let $A(\omega_j)$ (resp. $A(\bar{\eta}_j)$) be the operators of left exterior multiplication by ω_j (resp. $\bar{\eta}_j$). Then

$$(*) \qquad d = \sum_{i=1}^{N} A(\omega_i) \otimes 1 \otimes \omega(X_i) \otimes 1$$

and

$$(**) \qquad \bar{\partial} = \sum_{j=1}^{N'} 1 \otimes A(\bar{\eta}_j) \otimes \omega(\bar{Z}_j) \otimes 1.$$

We will give explicit formulas for $\omega(X_i)$ and $\omega(\bar{Z}_j)$ in § 7.

Remark. — We have intentionally abused notation here. The above operators d and $\bar{\partial}$ are the images of the usual operators d and $\bar{\partial}$ under the isomorphisms

$$(\mathscr{A}^i(D) \otimes \mathscr{A}^{0,j}(D') \otimes \mathscr{S}(V^n) \otimes \mathscr{L}_m)^{G \times \tilde{G}'} \to C^{i,j}.$$

Since these latter operators commute with $1 \otimes 1 \otimes \Theta \otimes 1$ (the notations Θ and θ_φ are defined below) we have formulas for $\varphi \in C^{i,j}$

$$d\theta_\varphi = \theta_{d\varphi} \quad \text{and} \quad \bar{\partial}\theta_\varphi = \theta_{\bar{\partial}\varphi}$$

where the d and $\bar{\partial}$ on the left are the usual operators from differential geometry and the operators d and $\bar{\partial}$ on the right are the operators defined by the formulas $(*)$ and $(**)$ above.

We will consider elements $\varphi \in C^{i,0}$ such that

(i) $d\varphi = 0$,
(ii) $\bar{\partial}[\varphi] = 0$.

Here $[\varphi]$ is the class of φ in the d-cohomology of $C^{i,j}$. We should emphasize that the equation $\bar{\partial}[\varphi] = 0$ means that there exists an element $\psi \in C^{i-1,1}$ such that

$$\bar{\partial}\varphi = d\psi.$$

Clearly such a φ gives rise to an element $[\varphi] \in H^i_{ct}(G, \mathscr{S}(V^n))^q_{\chi_m}$ using the notation of the introduction. We now want to explain how to compute the pairing $((\ ,\ ))$ and where the arithmetic lattice $\Gamma' \subset G'$ comes from.

We recall that there is a remarkable distribution Θ, the theta distribution, on $\mathscr{S}(V^n)$, which is the sum of Dirac delta distributions centered at points of L^n. Clearly, Θ is invariant under Γ so we have $\Theta \in \mathrm{Hom}_\Gamma(\mathscr{S}(V^n), \mathbf{C})$. Consequently if we define $\theta_\varphi(g', g)$ on $\tilde{G}' \times G$ by

$$\theta_\varphi(g', g) = \Theta(\omega(g')\,\rho(g)\,\varphi),$$

then $\theta_\varphi(g', g)$ descends in the G-variable to a closed differential form on M. As for the $\tilde{G}'$-variable we have the following theorem.

Theorem **3.1** (Weil [27]). — *There exists an arithmetic subgroup* $\Gamma' \subset G'$ *and a diagram*

$$\begin{array}{ccc} & & \widetilde{G}' \\ & \overset{s}{\nearrow} & \downarrow \\ \Gamma' & \longrightarrow & G' \end{array}$$

such that Θ *is invariant under* $\omega \mid s(\Gamma')$.

Corollary. — *We have* $\theta_\varphi(\gamma' g', g) = \theta_\varphi(g', g)$.

Here we identify Γ' with $s(\Gamma')$.

Thus in the $\widetilde{G}'$-variable, $\theta_\varphi(g', g)$ descends to a smooth (but non-holomorphic) section of the line bundle $\mathscr{L}_m$.

In fact we will refine the above construction to take into account the congruence condition of § 2. Let h and $\mathfrak{a}$ be as in § 2. We refine the theta distribution as follows

$$\Theta_h(x) = \sum_{\xi \in h + \mathfrak{a}L^n} \delta(x - \xi).$$

Given $\varphi \in \mathscr{S}(V^n)$ we now define $\theta_{\varphi, h}(g', g)$ on $C^\infty(\Gamma' \backslash \widetilde{G}') \times C^\infty(\Gamma \backslash G)$ by

$$\theta_{\varphi, h}(g', g) = \Theta_h(\omega(g')\, \varphi(g^{-1} x)).$$

Of course the groups Γ' and Γ leaving $\theta_{\varphi, h}(g', g)$ invariant are proper subgroups of those leaving $\theta_\varphi(g', g)$ invariant. Since we have fixed h and $\mathfrak{a}$ once and for all, we will drop the subscript h henceforth, hopefully without confusion. Hence $\theta_\varphi(g', g)$ means $\Theta_h(\omega(g')\, \varphi(g^{-1} x))$.

We then have the following for the pairing $((\ ,\ ))$ of the introduction.

Lemma **3.1.**

$$((\eta, [\varphi]))\,(g') = \int_M \eta \wedge \theta_\varphi(g', g).$$

We leave to the reader the task of proving that the pairing is well-defined on the cohomology level (this is just Stokes' Theorem [6]) and agrees with the pairing described in the introduction. We will use $\theta_\varphi(\eta)$ to denote the modular form defined by the integral above.

We will prove Theorems 2 and 3 of the introduction by computing the Fourier expansion of $\theta_\varphi(\eta)$ for a suitable φ (described in § 5). We now explain what this means. Note that $Sp(n, \mathbf{k})$ and $U(n, n)$ $(\mathbf{k}_0)$ have natural representations on $\mathbf{k}^{2n}$. Let $N_{\mathbf{k}}$ (resp. $N_{\mathbf{k}_0}$) denote the abelian unipotent subgroup of $Sp(n, \mathbf{k})$ (resp. $U(n, n)$ $(\mathbf{k}_0)$) consisting of those elements which leave fixed each of the first n standard basis vectors (recall that we are defining $U(n, n)$ $(\mathbf{k}_0)$ to be the isometry group of the standard split skew hermitian form over $\mathbf{k}$). We let N denote the corresponding subgroup of real points in G'. Then N is isomorphic to the sum of r copies of the space S of symmetric $n \times n$ real matrices (resp. $n \times n$ hermitian matrices) and $N \cap \Gamma'$ is a lattice $\mathscr{L}_1$ in S^r. We let $\mathscr{L}_1^*$ be the dual lattice to $\mathscr{L}_1$ in S^r for the bilinear form B given by

$$B(X, Y) = \operatorname{tr} XY.$$

Then $\theta_\varphi(\eta)$ has a Fourier expansion indexed by the elements of $\mathscr{L}_1^*$:

$$\theta_\varphi(\eta)\,(u + iv) = \sum_{\beta \in \mathscr{L}_1^*} a_\beta(v)\, e_*(2\beta u).$$

Here $\tau = u + iv$ is the decomposition of τ into real and imaginary parts (resp. hermitian and skew-hermitian parts) where τ is a point in $\mathfrak{h}_n^r$ (resp. $\mathbf{C}\mathfrak{h}_n^r$).

The following lemma will be useful to us in § 4.

Lemma **3.2.** — *Suppose η is a smooth differential form on $\Gamma\backslash D$ which is rapidly-decreasing and closed. Then there exists a smooth form η' on $\Gamma\backslash D$ which is compactly supported and closed such that*

$$\theta_\varphi(\eta) = \theta_\varphi(\eta').$$

Proof. — By Borel [3], there exists a smooth compactly supported closed form η' and a smooth rapidly decreasing differential form ξ such that

$$\eta - \eta' = d\xi,$$

whence

$$\theta_\varphi(\eta) - \theta_\varphi(\eta') = \theta_\varphi(d\xi) = \int_M d\xi \wedge \theta_\varphi.$$

But this latter integral is zero by Stokes' Theorem [6] since $d\xi \wedge \theta_\varphi$ and $\xi \wedge \theta_\varphi$ are both L^1 on $\Gamma\backslash D$. ∎

In the case that φ is a holomorphic class the calculation of the previous Fourier coefficients is greatly simplified by the following lemma.

Lemma **3.3.** — *If $[\varphi]$ is holomorphic and η is closed and rapidly decreasing, then $\theta_\varphi(\eta)$ is a holomorphic section of $\mathscr{L}_m$.*

Proof. — Assume $\bar\partial\varphi = d\psi$ with $\psi \in C^{i-1,1}$. Then we have

$$\bar\partial\theta_\varphi = d\theta_\psi.$$

We then have

$$\bar\partial\theta_\varphi(\eta) = \bar\partial\left(\int_M \eta \wedge \theta_\varphi\right) = \int_M \eta \wedge \bar\partial\theta_\varphi$$
$$= \int_M \eta \wedge d\theta_\psi = \int_M d(\eta \wedge \theta_\psi) = 0.$$

The last equality holds [7] because $\eta \wedge \theta_\varphi$ and $\eta \wedge \theta_\psi$ are both easily seen to be L^1 since η is rapidly decreasing. ∎

Corollary. — *We have* $a_\beta(v) = 0$ *unless* β *is totally positive semi-definite.*

Proof. — If $n \geqslant 2$ this is Koecher's Theorem [20], 4-04. If $n = 1$ the corollary follows because $\theta_\varphi(\eta)\,(\tau)$ is easily seen to have moderate growth. But if $a_\beta(\theta_\varphi(\eta))\,(v) \neq 0$ for some $\beta < 0$, then, since $\theta_\varphi(\eta)\,(\tau)$ is holomorphic, it would increase exponentially in v at ∞ (recall $\tau = u + iv$). ∎

Remark. — We have left to the reader the task of extending [20], 4-04, to the case in which $G' = U(n, n)$.

We will now evaluate the functions $a_\beta(\theta_\varphi(\eta))\,(\tau)$ in the case that φ is holomorphic and takes values in $\mathbf{S}(V^n)$ to obtain part (ii) of Theorem 1 in the introduction. We will consider only the case in which $G^{(1)} = O(p, q)$, the case of $G^{(1)} = U(p, q)$ is identical. We first recall the general results of [13] relating Fourier coefficients to periods. We write $\mathcal{Q}_\beta$ as a disjoint union of G-orbits

$$\mathcal{Q}_\beta = \coprod_{i \in I} \mathcal{O}_i.$$

Then we may write $\mathcal{Q}_\beta \cap (h + \mathfrak{a}L^n)$ as an (at most) countable disjoint union

$$\mathcal{Q}_\beta \cap (h + \mathfrak{a}L^n) = \coprod_{i \in I} (h + \mathfrak{a}L^n) \cap \mathcal{O}_i.$$

For $\mathcal{O}$ one of the above orbits we define

$$\theta_{\varphi, \mathcal{O}} = \sum_{X \in \mathcal{O}} \varphi(Z, X)$$

and
$$a_{\mathcal{O}} = \int_M \eta \wedge \theta_{\varphi, \mathcal{O}}.$$

Consequently we have

$$a_\beta(\theta_\varphi(\eta)) = \sum_{\mathcal{O}} a_{\mathcal{O}}.$$

In [13] we obtained a general formula for $a_{\mathcal{O}}$ in the case that $\mathcal{O}$ was non-degenerate. In this case we have the cycle $C_{\mathcal{O}} = \sum_{i=1}^{\ell} C_{U_i}$ of § 2. Then, as described in [13], we have a fiber bundle

$$p_{U_i} : D \to C_{U_i}.$$

We also define θ_{φ, U_i} by

$$\theta_{\varphi, U_i}(Z) = \sum_{X \in \mathcal{O} \cap U_i^n} \varphi(Z, X).$$

Remark. — In the above formula we have abused notation by omitting the dependence on τ: we consider that the τ-dependence is built into φ. We will continue to do this throughout, indicating the τ-dependence only where it becomes important. We recall that τ is a variable point in $\mathfrak{h}_n$ (resp. $\mathbf{C}\mathfrak{h}_n$).

We then have the following result which is an immediate consequence of Theorem 3.1 of [13] and Theorem 2.1 of [14] except for the case in which $G = SO(p, 1)$, $n = p - 1$. This case can be handled by a direct computation, see [18] where closely related integrals are computed.

Theorem **3.2.**

$$a_{\mathcal{O}} = \sum_{i=1}^{\ell} \int_{C_{U_i}} \eta \wedge (p_{U_i})_* \, \theta_{\varphi, U_i}.$$

Here, $(p_{U_i})_*$ denotes the operation of integrating over the fiber.

We now let D_{U_i} be as in § 2 and G'_{U_i} be the subgroup of G_{U_i} that acts trivially on U_i. Then $(p_{U_i})_* \, \theta_{\varphi, U_i}$ is the image of a G'_{U_i}-invariant form on D_{U_i}; however, G'_{U_i} does not act transitively on D_{U_i} unless $(\ , \ ) \mid U_i$ is positive definite.

We now specialize to the case in which β is totally positive definite. In this case $\mathcal{Q}_\beta$ consists of a single G-orbit and G'_{U_i} acts transitively on D_{U_i}. Hence $(p_{U_i})_* \, \theta_{\varphi, U_i} = c_i(\varphi)\,(v)$ pulls back to an invariant form on D_{U_i} in the usual sense—it is invariant under the connected component of the identity of the isometry group of D_{U_i}.

Lemma **3.4.** — *Suppose* φ *is a holomorphic class. Then*

$$a_\beta(\theta_\varphi(\eta))\,(v) = e_*(\beta i v) \sum_{i=1}^{k} \int_{C_{U_i}} \eta \wedge c_i(\varphi)\,(1).$$

Proof. — The function $\theta_\varphi(\eta)\,(\tau)$ is holomorphic. Hence by [20], 4-02,

$$a_\beta(\theta_\varphi(\eta))\,(v) = e_*(\beta i v)\, a_\beta(\theta_\varphi(\eta))\,(1). \quad \blacksquare$$

Remark. — Again we leave to the reader the problem of extending [20], 4-02, to the case in which $G' = U(n, n)$.

Finally the case in which β is totally positive semi-definite and $\mathcal{O} = \mathcal{O}_\beta^c$ is identical to the one just treated. In order to complete the proof of Theorem 1 of the introduction it remains to prove $a_{\mathcal{O}} = 0$ for the orbits $\mathcal{O} \subset \mathcal{Q}_\beta - \mathcal{Q}_\beta^c$. This will be accomplished in the next section, in case φ takes values in $\mathbf{S}(V^n)$. We observe that this problem arises only in the case $r = 1$, since, otherwise, there are no degenerate rational n-frames $X \in V^n$.

4. The vanishing of the contribution of the degenerate orbits to the Fourier coefficients of $\theta_\varphi(\eta)$

In this section we show that the frames in $\mathcal{Q}_\beta^d = \mathcal{Q}_\beta - \mathcal{Q}_\beta^c$ do not contribute to $a_\beta(\theta_\varphi(\eta))$. We will invoke Lemma 3.2 and assume η is compactly supported. We decompose $\mathcal{Q}_\beta$ further according to the dimension of $U = \text{span } X$; we define

$$\mathcal{Q}_{\beta, k} = \{ X \in \mathcal{Q}_\beta : X \in h + \mathfrak{a} L^n, \dim \text{span } X = k \}.$$

We have (for rank $\beta = t$)

$$\mathcal{Q}_\beta \cap (h + \mathfrak{a}L^n) = \coprod_{k=t}^{n} \mathcal{Q}_{\beta, k}.$$

We observe that $\mathcal{Q}_{\beta, t} = \mathcal{Q}_\beta^c$.

We define $\theta(Z; \beta, k)$ and $a_{\beta, k}(\theta_\varphi(\eta))$ by

$$\theta(Z; \beta, k) = \sum_{X \in \mathcal{Q}_{\beta, k}} \varphi(Z, X)$$

and

$$a_{\beta, k}(\theta_\varphi(\eta)) = \int_M \eta \wedge \theta(Z; \beta, k).$$

Theorem **4.1.** — *For any* $\varphi \in H^{\bullet}_{ct}(G, \mathscr{S}(V^n))$ *taking values in* $\mathbf{S}(V^n)$, *and any* $\eta \in H^{\bullet}_c(M, \mathbf{C})$, *and* β *positive semi-definite of rank* t, *we have*

$$a_{\beta, k}(\theta_\varphi(\eta)) = 0 \quad \textit{for } k > t.$$

Corollary. — *If* β *is positive semi-definite with rank* $\beta = t$, *then*

$$a_\beta(\theta_\varphi(\eta)) = a_{\beta, t}(\theta_\varphi(\eta)).$$

The proof of Theorem 4.1 will occupy the rest of this section. We put $r = k - t$, whence r is the dimension of the radical of $(\ ,\ ) \mid \operatorname{span} X$. Since there is a finite number of Γ-conjugacy classes in a single rational conjugacy class of parabolic subgroups of G, there exist a finite number of representatives $\{ R_1, R_2, \ldots, R_a \}$ for the Γ-orbits of totally isotropic rational subspaces of V of dimension r. If $X \in \mathcal{Q}_{\beta, k}$ we define $R(X) \in \mathrm{Gr}_r(V)$ to be the radical of $(\ ,\ ) \mid \operatorname{span} X$.

Let R be a rational isotropic subspace of V. We define $\mathcal{Q}_{\beta, R}$ by

$$\mathcal{Q}_{\beta, R} = \{ X \in h + \mathfrak{a}L^n : (X, X) = \beta, R(X) = R \}.$$

Here $R(X)$ denotes the radical of $(\ ,\ ) \mid U$. We define $\Gamma_R \subset \Gamma$ to be the stabilizer of R.

Then we have

$$\mathcal{Q}_{\beta, k} = \coprod_{i=1}^{a} \coprod_{\gamma \in \Gamma_{R_i} \backslash \Gamma} \mathcal{Q}_{\beta, \gamma^{-1} R_i}.$$

We now define $\theta(Z; \beta, R)$, a differential nq-form on $E = \Gamma_R \backslash D$, by

$$\theta(Z; \beta, R) = \sum_{X \in \mathcal{Q}_{\beta, R}} \varphi(Z, X).$$

Let $\mathscr{D}$ be a fundamental domain for the action of Γ on D.

Lemma **4.1.**

$$\int_M [\eta \wedge \theta(Z; \beta, k)] = \sum_{i=1}^{a} \int_{\Gamma_{R_i} \backslash D} [\eta \wedge \theta(Z; \beta, R_i)].$$

Proof.

$$\int_M [\eta \wedge \theta(Z; \beta, k)] = \int_M [\eta \wedge \sum_{i=1}^{a} \sum_{\gamma \in \Gamma_{R_i} \backslash \Gamma} \sum_{X \in \mathscr{Q}_{\beta, R_i}} \varphi(Z, \gamma^{-1} X)]$$

$$= \sum_{i=1}^{a} \int_{\mathscr{D}} [\eta \wedge \sum_{\gamma \in \Gamma_{R_i} \backslash \Gamma} \sum_{X \in \mathscr{Q}_{\beta, R_i}} \gamma^* \varphi(Z, X)]$$

$$= \sum_{i=1}^{a} \int_{\mathscr{D}} [\sum_{\gamma \in \Gamma_{R_i} \backslash \Gamma} \sum_{X \in \mathscr{Q}_{\beta, R_i}} \gamma^*(\eta \wedge \varphi(Z, X))]$$

$$= \sum_{i=1}^{a} \sum_{\gamma \in \Gamma_{R_i} \backslash \Gamma} \int_{\mathscr{D}} [\gamma^* \sum_{X \in \mathscr{Q}_{\beta, R_i}} \eta \wedge \varphi(Z, X)].$$

The previous exchange of summation and integration is valid because η is compactly supported. It is not valid for general η. We obtain

$$\int_M \eta \wedge \theta(Z; \beta, k) = \sum_{i=1}^{a} \sum_{\gamma \in \Gamma_{R_i} \backslash \Gamma} \int_{\gamma \mathscr{D}} \eta \wedge \sum_{X \in \mathscr{Q}_{\beta, R_i}} \varphi(Z, X)$$

$$= \sum_{i=1}^{a} \int_{\Gamma_{R_i} \backslash D} [\eta \wedge \sum_{X \in \mathscr{Q}_{\beta, R_i}} \varphi(Z, X)] = \sum_{i=1}^{a} \int_{\Gamma_{R_i} \backslash D} [\eta \wedge \theta(Z; \beta, R_i)]. \quad \blacksquare$$

Thus in order to prove Theorem 4.1 it suffices to prove the following theorem:

Theorem **4.1** *(bis). — Let* R *be a rational, non-zero, totally isotropic subspace of* V *and* η *a compactly supported form on* $\Gamma \backslash D$. *Then*

$$\int_{\Gamma_R \backslash D} \eta \wedge \theta(Z; \beta, R) = 0.$$

We let P be the parabolic subgroup of G defined by

$$P = \{ g \in G : gR = R \}.$$

We wish to parametrize P and then use this parametrization of P to parametrize the cusp $E = \Gamma_R \backslash D$.

We choose a rational totally isotropic subspace R′ which is dually paired with R and obtain a rational Witt decomposition

$$V = R + W + R'.$$

Let $U_1 = (R')^\perp \cap U$, whence $U_1 \subset (R + R')^\perp = W$ and $U = R + U_1$. We then split W further by choosing a maximal rational subspace $U_3 \subset W$ such that $(\ , \ ) \mid U_3$ is negative definite and $U_3 \subset U_1^\perp$. We define U′ to be the orthogonal complement of U_3 in W and U_2 to be the orthogonal complement of U_1 in U′. We obtain a rational orthogonal splitting

$$W = U_1 + U_2 + U_3.$$

We note that necessarily $\dim U_3 = q - r$, and $\dim U_1 = t$. We let $(\ ,\ )_0$ be a rational majorant of $(\ ,\ )$ such that the summands of the decomposition

$$V = R + U_1 + U_2 + U_3 + R'$$

are pairwise orthogonal for $(\ ,\ )_0$. We let Z_0 be the corresponding negative q-plane [2], 5.3. We choose a rational Witt basis $\{e_1, \ldots, e_r, w_1, \ldots, w_{m-2r}, f_1, f_2, \ldots, f_r\}$ for V with $e_1, \ldots, e_r \in R$ and $f_1, f_2, \ldots, f_r \in R'$ and $(e_i, f_j) = \delta_{ij}$. We may assume $e_1, e_2, \ldots, e_r \in L$. We define $a(t) \in P$ for $t \in \mathbf{R}$ by

$$\begin{aligned} a(t) \mid R &= e^t\, I_R, \\ a(t) \mid W &= I_W, \\ a(t) \mid R' &= e^{-t}\, I_R. \end{aligned}$$

Remark. — If we compactify D by taking its closure $\overline{D}$ in $\mathrm{Gr}_q(V)$ then $\overline{D}$ consists of the negative semi-definite q-planes and $\lim_{t\to\infty} a(t)\, Z_0$ is the negative semi-definite q-plane $R + U_3$.

We let $A = \{ a(t) : t \in \mathbf{R} \}$. Then we have the usual decomposition of P associated to the split torus A given by

$$P = NA\overline{M},$$

where $\overline{M}$ is the semi-simple part of the centralizer of A in P and N is the unipotent radical of P. We "parametrize" D by

$$\mu : N \times A \times S_{\overline{M}} \to D$$

according to

$$\mu(n, a, m) = namZ_0.$$

Here $S_{\overline{M}} = \overline{M}/K_{\overline{M}}$ is the symmetric space of $\overline{M}$ and $K_{\overline{M}}$ is the maximal compact subgroup of M that fixes Z_0. Clearly μ descends to a map

$$\mu' : \Gamma_R\backslash(N \times A \times S_{\overline{M}}) \to \Gamma_R\backslash D.$$

We let $p_1 : D \to \Gamma_R\backslash D$ and $p_2 : \Gamma_R\backslash D \to \Gamma\backslash D$ be the projections, whence $\pi = p_2 \circ p_1$. We let $t : D \to \mathbf{R}$ be defined so that $t \circ \mu(n, a(t), m) = t$. Then t descends to a function also denoted t from $\Gamma_R\backslash D$ to $\mathbf{R}$. Finally we define $a : \Gamma_R\backslash D \to \mathbf{R}_+$ by $a(Z) = e^{t(Z)}$. A level set $t^{-1}(c)$ of t separates $\Gamma_R\backslash D$ into two components, one $\{ Z : t(Z) \geqslant c \}$ of finite volume, the other $\{ Z : t(Z) \leqslant c \}$ of infinite volume. We will call the first set (for c large) the "cusp" corresponding to the rational isotropic subspace R.

Lemma **4.2.** — *Let* $C \subset M$ *be a compact subset. Then* $a \mid \pi^{-1}(C)$ *is bounded.*

Proof. — We consider the action of G on $\Lambda^r V$ and observe that $\Lambda^r L$ is a Γ-invariant lattice in $\Lambda^r V$. We abuse notation and let $(\ ,\ )_0$ denote the form on $\Lambda^r V$ induced by the above form $(\ ,\ )_0$ on V. Let $u_0 \in \Lambda^r V$ be given by $u_0 = e_1 \wedge e_2 \wedge \ldots \wedge e_r$. There exists a compact subset $\widetilde{C} \subset G$ such that $\pi^{-1}(C) \subset \Gamma\widetilde{C}$. Indeed we choose $\widetilde{C}$ to be the inverse image of C under π restricted to the closure of a fundamental domain for Γ in G. We next observe that the formula

$$a(p_1(g))^{2r} = \frac{1}{(g^{-1}u_0, g^{-1}u_0)_0}$$

is immediate upon writing $g = na(t)\,mk$ and noting

$$(na(t)\,m)^{-1}u_0 = e^{-rt}u_0.$$

Thus it is sufficient to prove that $f(g) = \dfrac{1}{(g^{-1}u_0, g^{-1}u_0)_0}$ is bounded on the subset $\Gamma\widetilde{C} \subset G$. Therefore it suffices to prove there exists $\varepsilon > 0$ such that

$$\min\{(c^{-1}\gamma^{-1}u_0, c^{-1}\gamma^{-1}u_0) : c \in \widetilde{C}, \gamma \in \Gamma\} \geqslant \varepsilon.$$

Now the set of quadratic forms $\{(c^{-1})^*(\ ,\ )_0 : c \in \widetilde{C}\}$ is a compact set, whence there exists $\eta > 0$ independent of C such that

$$(c^{-1})^*(\ ,\ )_0 \geqslant \eta(\ ,\ )_0.$$

Now let β be the lower bound of $(\ ,\ )_0$ on the non-zero vectors in the lattice $\Lambda^r L$. Then clearly the above inequality holds with $\varepsilon = \eta\beta$. ■

Corollary. — *For t sufficiently large*

$$(\pi^*\eta)(\mu(n, a(t), m)) = 0.$$

Proof. — The form η has compact support C on M so the support of $\pi^*\eta$ is contained in $\pi^{-1}(C)$. ■

In the appendix to this paper we will prove the following lemma showing that $||\theta(Z; \beta, R)||$ decays rapidly as $t \to -\infty$ (the other end of $E - t^{-1}(1)$ from the cusp). Here $||\ \ ||$ denotes the pointwise norm for the induced Riemannian metric on E.

Lemma 4.3. — *There exist positive constants* C *and* ε *(depending on* T*) such that for all t satisfying* $-\infty < t \leqslant T$ *we have*

$$||\theta_\varphi||(\mu(n, a(t), m)) \leqslant Ce^{-\varepsilon e^{-2t}}.$$

Corollary. — *The form* $\eta \wedge \theta(Z; \beta, R)$ *is integrable on* $\Gamma_R\backslash D$.

Proof. — The volume form of $\Gamma_R\backslash D$ is up to a sign of the form $e^{-\beta t}\,\mathrm{vol}_N \wedge dt \wedge \mathrm{vol}_{S_M}$ for some positive constant β. Moreover, since η has compact support on M, the Riemannian norm of η is bounded and we have as a consequence of the above inequality the following (with C′ a positive constant):

$$||\eta \wedge \theta_\varphi||(\mu(n, a(t), m)) \leqslant C'\,e^{-\varepsilon e^{-2t}}.$$

The corollary follows. ■

We can now prove Theorem 4.1 *(bis)*. Let α_λ, $\lambda \in \mathbf{R}_+$, denote the multiplicative one-parameter group of diffeomorphisms of E given by

$$\alpha_\lambda(na(t)\, mZ_0) = na(t + \log \lambda)\, mZ_0.$$

We note the formula

$$a(\alpha_\lambda(Z)) = \lambda a(Z).$$

We define a form ψ on E of degree one less than φ by

$$\psi(Z) = \int_0^1 \iota_{a\frac{\partial}{\partial a}} \alpha_\lambda^* \theta(Z; \beta, R) \frac{d\lambda}{\lambda}.$$

Here $\iota_{a\frac{\partial}{\partial a}}$ denotes interior multiplication by the vector field $a\dfrac{\partial}{\partial a}$ which generates α_λ. We let $\mathscr{L}_{a\frac{\partial}{\partial a}}$ denote the operation of Lie derivation of differential forms by $a\dfrac{\partial}{\partial a}$. We claim that $d\psi = \theta(Z; \beta, R)$. Indeed, using the Cartan formula

$$\mathscr{L}_{a\frac{\partial}{\partial a}} = \iota_{a\frac{\partial}{\partial a}} \circ d + d \circ \iota_{a\frac{\partial}{\partial a}}$$

and recalling that $\alpha_\lambda^* \theta(Z; \beta, R)$ is closed, we have

$$\begin{aligned}
d\psi &= d\iota_{a\frac{\partial}{\partial a}} \int_0^1 \alpha_\lambda^* \theta(Z; \beta, R) \frac{d\lambda}{\lambda} \\
&= \mathscr{L}_{a\frac{\partial}{\partial a}} \left(\int_0^1 \alpha_\lambda^* \theta(Z; \beta, R) \frac{d\lambda}{\lambda} \right) \\
&= \frac{d}{d\mu} \left(\alpha_\mu^* \int_0^1 \alpha_\lambda^* \theta(Z; \beta, R) \frac{d\lambda}{\lambda} \right) \bigg|_{\mu=1} \\
&= \frac{d}{d\mu} \left(\int_0^1 \alpha_{\mu\lambda}^* \theta(Z; \beta, R) \frac{d\lambda}{\lambda} \right) \bigg|_{\mu=1} \\
&= \frac{d}{d\mu} \left(\int_0^\mu \alpha_\lambda^* \theta(Z; \beta, R) \frac{d\lambda}{\lambda} \right) \bigg|_{\mu=1} \\
&= (\alpha_\mu^* \theta(Z; \beta, R))|_{\mu=1} = \theta(Z; \beta, R).
\end{aligned}$$

Thus $d\psi = \theta(Z, \beta, R)$.

We now prove that ψ has rapid decrease at $t \to -\infty$ (or $a \to 0$). Indeed, let $T \in \mathbf{R}$ be given and let $E_T \subset E$ be defined by

$$E_T = \{ Z \in E \mid a(Z) \leqslant T \}.$$

We first claim that if ν is a smooth differential k-form on E_T, then for all $\lambda \in (0, 1]$ and $Z \in E_T$ we have

$$\| \alpha_\lambda^* \nu(Z) \| \leqslant \alpha_\lambda^* \| \nu(Z) \|.$$

Indeed, it suffices to prove the corresponding inequality when all the terms are lifted to P. Let (,) denote a left P-invariant, right $K \cap P$-invariant Riemannian metric inducing the given Riemannian metric on E. Choose a basis $\{ \omega_i \}$ for the left invariant k-forms on P which is orthonormal at the identity e on P (and hence at all points of P by invariance). We may further assume that $\{ \omega_i |_e \}$ consists of eigenvectors for the coadjoint action σ of A on $\Lambda^k T_e^*(P)$. Hence there are homomorphisms $\chi_i : \mathbf{R}_+ \to \mathbf{R}_+$ such that

$$\sigma(a(\log \lambda)^{-1}) \, \omega_i |_e = \chi_i(\lambda) \, \omega_i |_e.$$

Suppose now that $\tilde{\nu}$, the lift of ν to P, satisfies $\tilde{\nu} = \Sigma_i f_i \, \omega_i$. We then have

$$|| \tilde{\nu} ||^2 = \sum_i f_i^2.$$

We next observe that the lift (again denoted by α_λ) of α_λ to P coincides with right multiplication by $a(\log \lambda)$, whence

$$\alpha_\lambda^* \, \omega_i = \chi_i(\lambda) \, \omega_i.$$

We obtain

$$|| \alpha_\lambda^* \, \tilde{\nu} ||^2 = \sum_i \chi_i(\lambda)^2 \, \alpha_\lambda^* f_i^2,$$

$$\alpha_\lambda^* \, || \tilde{\nu} ||^2 = \sum_i \alpha_\lambda^* f_i^2.$$

Finally we note that the eigenvalues of Ad $a(\log \lambda)^{-1}$ on $T_e(P)$ are 1, λ^{-1} and λ^{-2}, whence the eigenvalues on $T_e^*(P)$ are 1, λ and λ^2, and, consequently, the functions $\chi_i(\lambda)$ are all majorized by 1 for $\lambda \in (0, 1]$. The claim follows.

We may now estimate $|| \psi ||$ on E_T. We have

$$\begin{aligned} || \psi || &\leqslant \int_0^1 || \iota_{a \frac{\partial}{\partial a}} \alpha_\lambda^* \, \theta(Z; \beta, R) || \frac{d\lambda}{\lambda} \\ &\leqslant \int_0^1 || \alpha_\lambda^* \, \theta(Z; \beta, R) || \frac{d\lambda}{\lambda} \\ &\leqslant \int_0^1 \alpha_\lambda^* \, || \theta(Z; \beta, R) || \frac{d\lambda}{\lambda} \\ &\leqslant C \int_0^1 e^{-\varepsilon(\lambda a)^{-2}} \frac{d\lambda}{\lambda} \\ &\leqslant C e^{-\frac{\varepsilon}{2} a^{-2}} \int_0^1 e^{-\frac{\varepsilon}{2}(T\lambda)^{-2}} \frac{d\lambda}{\lambda} \\ &\leqslant C' \, e^{-\frac{\varepsilon}{2} a^{-2}}. \end{aligned}$$

It is now clear that ψ has rapid decrease as $t \to -\infty$. Combining the corollary to Lemma 4.2 with the above we see as in the corollary to Lemma 4.3 that $\eta \wedge \psi$ is integrable on E and satisfies

$$d(\eta \wedge \psi) = \eta \wedge \theta(Z; \beta, R).$$

Since E is complete and $\eta \wedge \theta(Z; \beta, R)$ is integrable with an integrable primitive, it follows from Stokes Theorem for non-compact manifolds [6] that

$$\int_E \eta \wedge \theta(Z; \beta, R) = 0. \quad \blacksquare$$

5. Construction of holomorphic Schwartz classes

In this section we will construct holomorphic cohomology classes (for $q \geqslant 1$) in $H^{\bullet}(G, \mathscr{S}(V^n))$ that take values in $\mathbf{S}(V^n)$. In this computation the compact factors of G play no role as we now explain. Using the notation established in § 2, we write

$$G \cong G^{(1)} \times H$$

and

$$V = V^{(1)} \oplus W,$$

where H is compact and $(\, , \,)|_W$ is positive definite. We let $\varphi' \in \mathscr{S}(W^n)$ be the function defined by

$$\varphi'(x_1, \ldots, x_n) = e^{-\pi \sum_{i=1}^n (x_i, x_i)}.$$

Then multiplication by φ' gives an isomorphism (for all i)

$$\mu : H^i(G^{(1)}, \mathscr{S}((V^{(1)})^n)) \to H^i(G, \mathscr{S}(V^n)).$$

Next observe that $Sp(n, \mathbf{R})^\sim$ (resp. $U(n, n)^\sim$) operates on the domain of μ and $Sp(n, \mathbf{R})^{\sim r}$ (resp. $U(n, n)^{\sim r}$) operates on the target of μ. Also, μ carries holomorphic classes for $Sp(n, \mathbf{R})^\sim$ (resp. $U(n, n)^\sim$) to holomorphic classes for $Sp(n, \mathbf{R})^{\sim r}$ (resp. $U(n, n)^{\sim r}$). This is because $\varphi' \in H^0(H, \mathscr{S}(W^n))$ is holomorphic for the induced action of $(Sp(n, \mathbf{R})^\sim)^{r-1}$ (resp. $(U(n, n)^\sim)^{r-1}$). Thus for considerations of the above continuous cohomology groups it suffices to assume that $r = 1$. We leave it to the reader to check that the transformation laws of φ_{nq}^+ under $U(n)^\sim$ and of $\varphi_{nq, nq}^+$ under $(U(n) \times U(n))^\sim$ given in (i) and (ii) of Theorem 5.2 match up with the transformation law of φ' under $(U(n)^\sim)^{r-1}$ (resp. $((U(n) \times U(n))^\sim)^{r-1}$).

Thus for the next four sections we will assume that $r = 1$ and we will construct holomorphic classes (for $q \geqslant 1$)

$$\varphi_{nq}^+ \in H_{ct}^{nq}(O(p, q), \mathscr{S}(V^n)),$$

$$\varphi_{nq, nq}^+ \in H_{ct}^{nq, nq}(U(p, q), \mathscr{S}(V^n)),$$

which take values in the polynomial Fock space $\mathbf{S}(V^n)$.

We begin by choosing bases for $\mathfrak{p}$ and $\mathfrak{p}^*$. First we treat the orthogonal case. Let $\{ v_1, v_2, \ldots, v_m \}$ be a properly oriented (see § 2) orthogonal basis for V chosen such that

$$(v_\alpha, v_\alpha) = 1, \quad 1 \leqslant \alpha \leqslant p \quad \text{and} \quad (v_\mu, v_\mu) = -1, \quad p + 1 \leqslant \mu \leqslant p + q.$$

We may assume that Z_0 is the span of $\{v_{p+1}, \ldots, v_{p+q}\}$ and that $\{v_{p+1}, \ldots, v_{p+q}\}$ is a properly oriented basis for Z_0. We recall that there is on $O(V)$-equivariant isomorphism $\rho : \Lambda^2 V \to \mathfrak{o}(V)$ given by

$$\rho(v \wedge v')\, v'' = (v, v'')\, v' - (v', v'')\, v.$$

We define a basis $\{X_{ij} : 1 \leqslant i < j \leqslant m\}$ for $\mathfrak{g}$ by $X_{ij} = \rho(v_i \wedge v_j)$ and we let $\{\omega_{ij}\} \subset \mathfrak{g}^*$ be the dual basis to $\{X_{ij}\}$. We will follow the convention that α, β will denote indices between 1 and p and μ, ν will denotes indices between $p+1$ and $p+q$. With this convention the set of $\{X_{\alpha\beta}, X_{\mu\nu}\}$ are a basis for $\mathfrak{k}$ and $\{X_{\alpha\mu}\}$ for $\mathfrak{p}$. The set $\{\omega_{\alpha\mu}\}$ is a basis for $\mathfrak{p}^*$.

We next treat the unitary case. In this case V is an m-dimensional complex vector space and we choose an orthogonal **C**-basis $\{v_1, \ldots, v_m\}$ such that

$$(v_\alpha, v_\alpha) = 1, \quad 1 \leqslant \alpha \leqslant p \quad \text{and} \quad (v_\mu, v_\mu) = -1, \quad p+1 \leqslant \mu \leqslant p+q.$$

We recall that there is a **C**-linear isomorphism $\rho : V \otimes_{\mathbf{C}} \overline{V} \to \mathfrak{gl}(V)$ given by

$$\rho(v \otimes \bar{v}')\, v'' = (v'', v')\, v.$$

Here, $\bar{v}'$ denotes the image of v' by the natural map $V \to \overline{V}$.

The anti-linear map $v \otimes \bar{v}' \mapsto -v' \otimes \bar{v}$ induces the real structure on $V \otimes_{\mathbf{C}} \overline{V}$ that corresponds to the conjugation of $\mathfrak{gl}(V)$ relative to $\mathfrak{u}(V)$. Here we are assuming (,) is **C**-linear on V in the first argument and **C**-anti-linear on V in the second. We define elements $X_{jk} \in \mathfrak{g}$, for $1 \leqslant j < k \leqslant m$, and $Y_{jk} \in \mathfrak{g}$, for $1 \leqslant j \leqslant k \leqslant m$, by

$$X_{jk} = -\rho(v_j \otimes \bar{v}_k - v_k \otimes \bar{v}_j), \qquad Y_{jk} = -\rho(i(v_j \otimes \bar{v}_k + v_k \otimes \bar{v}_j)).$$

Then $\{X_{jk} : 1 \leqslant j < k \leqslant m\} \cup \{Y_{jk} : 1 \leqslant j \leqslant k \leqslant m\}$ is a basis for $\mathfrak{g}$. We let $\{\omega_{jk}, \omega'_{jk}\}$ be the dual basis for $\mathfrak{g}^*$. Thus $\{\omega_{\alpha\mu}, \omega'_{\alpha\mu} : 1 \leqslant \alpha \leqslant p, p+1 \leqslant \mu \leqslant p+q\}$ is a basis for $\mathfrak{p}^*$. We define

$$\xi_{\alpha\mu} = \omega_{\alpha\mu} + i\omega'_{\alpha\mu}, \quad 1 \leqslant \alpha \leqslant p, \quad p+1 \leqslant \mu \leqslant p+q.$$

Then $\{\xi_{\alpha\mu}\}$ is a basis for the horizontal Maurer-Cartan forms on G of type (1, 0).

The coordinates $\{(x_i) : 1 \leqslant i \leqslant m\}$ in V or complex coordinates $\{(z_i) : 1 \leqslant i \leqslant m\}$ in the unitary case relative to the above bases $\{v_1, \ldots, v_m\}$ satisfy

$$(x, x) = \sum_{\alpha=1}^{p} x_\alpha^2 - \sum_{\mu=p+1}^{p+q} x_\mu^2$$

or (in the unitary case)

$$(x, x) = \sum_{\alpha=1}^{p} |z_\alpha|^2 - \sum_{\mu=p+1}^{p+q} |z_\mu|^2.$$

In the two cases we define φ_0, the Gaussian on V, by

$$\varphi_0(x) = e^{-\pi \Sigma_{i=1}^m x_i^2} \quad \text{or} \quad \varphi_0(x) = e^{-\pi \Sigma_{i=1}^m |z_i|^2}.$$

In the orthogonal case we define the Howe operator ([12])

$$D^+ : \Lambda^* \mathfrak{p}^* \otimes \mathscr{S}(V) \to \Lambda^{*+q} \mathfrak{p}^* \otimes \mathscr{S}(V)$$

by

$$D^+ = \frac{1}{2^q} \prod_{\mu=p+1}^{m} \left\{ \sum_{\alpha=1}^{p} \left[A_{\alpha\mu} \otimes \left(x_\alpha - \frac{1}{2\pi} \frac{\partial}{\partial x_\alpha} \right) \right] \right\}.$$

Here, $A_{\alpha\mu}$ denotes left multiplication by $\omega_{\alpha\mu}$.

We define

$$\varphi_q^+ = \mathrm{D}^+ \varphi_0.$$

In the unitary case we define

$$\mathrm{D}^+ = \frac{1}{2^{2q}} \prod_{\mu=p+1}^{m} \left\{ \sum_{\alpha=1}^{p} \left[\mathrm{A}_{\alpha\mu} \otimes \left(\bar{z}_\alpha - \frac{1}{\pi} \frac{\partial}{\partial z_\alpha} \right) \right] \right\}.$$

Here $\partial/\partial z_\alpha = \dfrac{1}{2}\left[\dfrac{\partial}{\partial x_\alpha} - i\,\dfrac{\partial}{\partial y_\alpha}\right]$ and $\mathrm{A}_{\alpha\mu}$ denotes the left multiplication by the element $\xi_{\alpha\mu}$ in $(\mathfrak{p}^*)^+$ described above. In the unitary case we define

$$\varphi_{q,q}^+ = \mathrm{D}^+ \, \overline{\mathrm{D}}^+ \, \varphi_0.$$

We have the following theorem with $m = p + q$.

Theorem **5.1.** — (i) φ_q^+ *and* $\varphi_{q,q}^+$ *are closed.*

(ii) φ_q^+ *transforms under* $\widetilde{\mathrm{U}}(1) \subset \widetilde{\mathrm{Sp}}_1(\mathbf{R})$ *according to* $\det^{m/2}$.

(iii) $\varphi_{q,q}^+$ *transforms under* $\mathrm{U}(1) \times \mathrm{U}(1) \subset \mathrm{U}(1, 1)$ *according to* $\det_+^m \otimes \det_-^{-m}$.

(iv) φ_q^+ *and* $\varphi_{q,q}^+$ *are holomorphic and take values in* $\mathbf{S}(\mathrm{V}^n)$.

Remark. — In (ii) (resp. (iii)) det denotes the identity character of the group U(1) (resp. $\det_+$ and $\det_-$ are the projections on the first and second factors of $\mathrm{U}(1) \times \mathrm{U}(1)$). The group $\widetilde{\mathrm{U}}(1)$ coincides with the connected covering group of U(1) such that the pullback of det admits a square root $(\det)^{\frac{1}{2}}$, [16].

Proof. — Parts (i), (ii) and (iii) of Theorem 5.1 are proved in [12]. The result that φ_q^+ is a holomorphic class is hard and will be proved in § 8. Assuming this result we prove that $\varphi_{q,q}^+$ is holomorphic by a " see-saw pair " argument.

We have a " see-saw pair "

$$\begin{tikzcd} \mathrm{U}(1,1) \arrow[d, dash] \arrow[dr, dash] & \mathrm{O}(2p,2q) \arrow[d, dash] \arrow[dl, dash] \\ \widetilde{\mathrm{Sp}}_1(\mathbf{R}) & \mathrm{U}(p,q) \end{tikzcd}$$

All four groups are subgroups of $\mathrm{Sp}_{2m}(\mathbf{R})$ and operate on $\mathscr{S}(\mathrm{V})$ by the appropriate restrictions of the oscillator representation. Moreover, we have the easily verified fact, [12], § 6, that the image of φ_{2q}^+ under the natural restriction map of relative Lie algebra cohomology complexes is $\varphi_{q,q}^+$. Now we have seen that $[\varphi_q^+]$ is annihilated by the anti-holomorphic tangent space of $\mathfrak{h}_1 = \mathrm{Sp}_1(\mathbf{R})/\mathrm{U}(1)$ of the identity coset. But the anti-holomorphic tangent spaces of $\mathrm{U}(1, 1)/\mathrm{U}(1) \times \mathrm{U}(1)$ and $\mathrm{Sp}_1(\mathbf{R})/\mathrm{U}(1)$ at the identity coset coincide since the symmetric spaces themselves coincide. ■

Thus we obtain holomorphic classes

$$\varphi_q^+ \in \mathrm{H}_{ct}^q(\mathrm{O}(p, q), \mathscr{S}(\mathrm{V})) \quad \text{and} \quad \varphi_{q,q}^+ \in \mathrm{H}_{ct}^{q,q}(\mathrm{U}(p, q), \mathscr{S}(\mathrm{V})).$$

There are corresponding classes φ_p^- and $\varphi_{p,p}^-$ obtained by replacing D^+ by the operator D^- given by

$$D^- = \frac{1}{2^p} \prod_{\alpha=1}^{p} \left\{ \sum_{\mu=p+1}^{m} \left[A_{\alpha\mu} \otimes \left(x_\mu - \frac{1}{2\pi} \frac{\partial}{\partial x_\mu} \right) \right] \right\},$$

or

$$D^- = \frac{1}{2^{2p}} \prod_{\alpha=1}^{p} \left\{ \sum_{\mu=p+1}^{m} \left[A_{\alpha\mu} \otimes \left(\bar{z}_\alpha - \frac{1}{\pi} \frac{\partial}{\partial z_\mu} \right) \right] \right\}.$$

To construct more classes we note that we have an exterior product to be denoted $\wedge$

$$H_{ct}^{\ell}(G, \mathscr{S}(V)) \otimes H_{ct}^{r}(G, \mathscr{S}(V)) \to H_{ct}^{\ell+r}(G, \mathscr{S}(V^2))$$

given by the usual formula for exterior product using the natural map from $\mathscr{S}(V) \otimes \mathscr{S}(V)$ to $\mathscr{S}(V^2)$.

We find the following results for the n-fold exterior power of the basic class. Recall $m = p + q$.

Theorem **5.2.** — (i) $\varphi_{nq}^+ = \varphi_q^+ \wedge \ldots \wedge \varphi_q^+$ *transforms under* $\widetilde{U}(n)$ *according to* $(\det)^{m/2}$.
(ii) $\varphi_{nq,nq}^+ = \varphi_{q,q}^+ \wedge \ldots \wedge \varphi_{q,q}^+$ *transforms under* $U(n) \times U(n)$ *according to* $\det_+^m \otimes \det_-^{-m}$.
(iii) φ_{nq}^+ *and* $\varphi_{nq,nq}^+$ *are holomorphic and take values in* $\mathbf{S}(V^n)$.

Proof. — Parts (i) and (ii) of the theorem are proved in [12]. It remains to prove (iii). Once again our proof will assume that φ_q^+ is a holomorphic class. We first prove that φ_{nq}^+ is a holomorphic class again by considering the appropriate see-saw pair.

We consider the see-saw pair

$$\begin{tikzcd}[column sep=huge]
O(p,q) \times \ldots \times O(p,q) \arrow[drr, dash] & & \widetilde{Sp}_n(\mathbf{R}) \arrow[d, dash] \arrow[dll, dash] \\
O(p,q) \arrow[u, dash, "\Delta"] & & \widetilde{Sp}_1(\mathbf{R}) \times \ldots \times \widetilde{Sp}_1(\mathbf{R}).
\end{tikzcd}$$

Here both products are n-fold products, Δ denotes the diagonal embedding and we have decomposed the $2n$-dimensional symplectic space into an orthogonal sum of symplectic planes. All four groups in the see-saw are subgroups of $\widetilde{Sp}_{nm}(\mathbf{R})$ and act on $\mathscr{S}(V^n)$ via the appropriate restrictions of the oscillator representation of $\widetilde{Sp}_{nm}(\mathbf{R})$. Now let D be the symmetric space of $O(p, q)$ and $\Delta : D \to D^n$ be the diagonal mapping. Then as explained in [12], p. 376, we have

$$\varphi_{nq}^+ = \Delta^*(p_1^* \varphi_q^+ \wedge \ldots \wedge p_n^* \varphi_q^+),$$

where $p_j : D^n \to D$ is projection on the j-th factor. From this formula it is immediate that the class φ_{nq}^+ is annihilated by the anti-holomorphic tangent space $\mathfrak{p}_0^-$ of

$$Sp_1(\mathbf{R})/U(1) \times \ldots \times Sp_1(\mathbf{R})/U(1)$$

at the identity coset. On the other hand, the class φ^+_{nq} is an eigenclass of $\widetilde{\mathrm{U}}(n)$, whence φ^+_{nq} is annihilated by the Ad $\widetilde{\mathrm{U}}(n)$ orbit of $\mathfrak{p}^-_0$ inside $\mathfrak{p}^-$, the anti-holomorphic tangent space of $\mathrm{Sp}_n(\mathbf{R})/\mathrm{U}(n)$. But the linear span of this orbit is all of $\mathfrak{p}^-$.

The proof that $\varphi^+_{nq,\,nq}$ is a holomorphic class is analogous using the see-saw pair

$$\begin{tikzcd}
\mathrm{U}(p, q) \times \ldots \times \mathrm{U}(p, q) \arrow[d, dash, "\Delta"'] \arrow[drr, dash] & & \mathrm{U}(n, n) \arrow[d, dash] \arrow[dll, dash] \\
\mathrm{U}(p, q) & & \mathrm{U}(1, 1) \times \ldots \times \mathrm{U}(1, 1).
\end{tikzcd}$$

Henceforth we will use a single symbol φ to denote one of the classes (for varying n and q) in

$$\mathrm{H}^{nq}_{ct}(\mathrm{O}(p, q) \times \prod_{\alpha=2}^{r} \mathrm{O}_m(\mathbf{R}), \mathscr{S}(\mathrm{V}^n))$$

(resp. $$\mathrm{H}^{nq}_{ct}(\mathrm{U}(p, q) \times \prod_{\alpha=2}^{r} \mathrm{U}(m), \mathscr{S}(\mathrm{V}^n)))$$

given by $\varphi = \varphi^+_{nq} \otimes \varphi_0 \otimes \ldots \otimes \varphi_0$ (resp. $\varphi = \varphi^+_{nq,\,nq} \otimes \varphi_0 \otimes \ldots \otimes \varphi_0$).

5. The Infinitesimal Fock Model

Let W be a vector space over $\mathbf{R}$ with a non-degenerate skew-symmetric form $\langle\ ,\ \rangle$ and let J_0 be a positive definite complex structure (i. e. the form $(\ ,\ )$ on W defined by $(w_1, w_2) = \langle \mathrm{J}_0 w_1, w_2 \rangle$ is positive definite symmetric) on W. Let $\{e_1, e_2, \ldots, e_n, f_1, f_2, \ldots, f_n\}$ be a symplectic basis for W such that

$$\mathrm{J}_0 e_i = f_i \quad \text{and} \quad \mathrm{J}_0 f_i = -e_i \quad \text{for } 1 \leqslant i \leqslant n.$$

We may decompose $\mathrm{W} \otimes \mathbf{C}$ according to

$$\mathrm{W} \otimes \mathbf{C} = \mathrm{W}' + \mathrm{W}''$$

where W' is the $+i$ eigenspace of J_0 and W'' is the $-i$ eigenspace of J_0. We define complex bases $\{w'_1, w'_2, \ldots, w'_n\}$ and $\{w''_1, w''_2, \ldots, w''_n\}$ for W' and W'' respectively by

$$w'_j = e_j - if_j \quad \text{and} \quad w''_j = e_j + if_j \quad \text{for } 1 \leqslant j \leqslant n.$$

Then W' and W'' are dually paired Lagrangian subspaces of $\mathrm{W} \otimes \mathbf{C}$ and we have

$$\langle w'_j, w''_k \rangle = 2i\delta_{jk}.$$

We will identify the Lie algebra $\mathfrak{sp}(\mathrm{W})$ with $\mathrm{S}^2(\mathrm{W})$ using the form $\langle\ ,\ \rangle$ as follows. Let $x \circ y \in \mathrm{S}^2(\mathrm{W})$ be defined by

$$x \circ y = x \otimes y + y \otimes x.$$

We then define $\varphi : \mathrm{S}^2(\mathrm{W}) \to \mathfrak{sp}(\mathrm{W})$ by

$$\varphi(x \circ y)\,(z) = \langle x, z \rangle y + \langle y, z \rangle x.$$

We use the symbol φ for the induced map on the complexifications as well. The complex structures J_0 induces a linear map DJ_0 (the derivation extension of J_0) on all tensor spaces of W. In particular we have

$$DJ_0(x \circ y) = (J_0 x) \circ y + x \circ (J_0 y).$$

The operator DJ_0 has eigenvalues $2i$, $-2i$, and 0 on $S^2(W \otimes \mathbf{C})$ with eigenspaces $S^2 W'$, $S^2 W''$ and $W' \otimes W''$. It acts on $\mathfrak{sp}(W)_{\mathbf{C}}$ (using φ) and the zero eigenspace is the Lie algebra $\mathfrak{u}(n)$ of the maximal compact subgroup $U(n) \subset Sp(W)$ consisting of those elements which centralize J_0. Then φ identifies $\mathfrak{p}'$ with the -1-eigenspace of $S^2 J_0$ on $S^2 W$ and $(-DJ_0)$ induces an almost complex structure on $\mathfrak{p}'$ which is $U(n)$-invariant and consequently extends to an $Sp(W)$ invariant almost complex structure on the Siegel space $\mathfrak{h}_n$ which is necessarily integrable. The holomorphic and anti-holomorphic tangent spaces at the base-point J_0 are given respectively (in terms of φ) by

$$\mathfrak{p}^+ = S^2(W'') \quad \text{and} \quad \mathfrak{p}^- = S^2(W').$$

We have chosen the complex structure $-DJ_0$ in order to agree with the usual complex structure which $\mathfrak{h}_n$ inherits as an open subset of the n by n symmetric matrices.

We now construct a one-parameter family of representations ω_λ of $\mathfrak{sp}(W \otimes \mathbf{C})$ on $S^*(W')^*$, the symmetric algebra of the dual of W'. We observe that W' and W'' are dually paired by $\langle\ ,\ \rangle$ and consequently it suffices to construct a one-parameter family of representations on $S^*(W'')$.

We first define a one-parameter family of algebras, the Weyl algebras $\mathscr{W}_\lambda$, associated to the pair W, $\langle\ ,\ \rangle$. We define $\mathscr{W}_\lambda$ to be the quotient of the tensor algebra $\bigotimes (W \otimes \mathbf{C})$ of the complexification of W by the ideal $\mathscr{I}$ generated by the elements $x \otimes y - y \otimes x - \lambda \langle x, y \rangle 1$. We let $p : \bigotimes (W \otimes \mathbf{C}) \to \mathscr{W}_\lambda$ be the quotient map. Clearly $p(\bigotimes W') = S^*(W')$ and $p(\bigotimes W'') = S^*(W'')$. We observe that $\mathscr{W}_\lambda$ has a filtration F^* inherited from the grading of $\bigotimes (W \otimes \mathbf{C})$ and that

$$[F^p \mathscr{W}_\lambda, F^q \mathscr{W}_\lambda] \subset F^{p+q-2} \mathscr{W}_\lambda.$$

Thus $F^2 \mathscr{W}_\lambda$ is a Lie algebra and we have a split extension of Lie algebras

$$F^1 \mathscr{W}_\lambda \to F^2 \mathscr{W}_\lambda \to \mathfrak{sp}(W \otimes \mathbf{C}),$$

where the splitting map $j : \mathfrak{sp}(W \otimes \mathbf{C}) = S^2(W \otimes \mathbf{C}) \to F^2 \mathscr{W}_\lambda$ is given by

$$j(x \circ y) = -\frac{1}{2\lambda} p(x \circ y) = -\frac{1}{2\lambda} [p(x)\, p(y) + p(y)\, p(x)].$$

We now let $\mathscr{J}$ denote the left ideal in $\mathscr{W}_\lambda$ generated by W'. Then $\mathscr{W}_\lambda / \mathscr{J}$ is a $\mathscr{W}_\lambda$-module and a fortiori an $\mathfrak{sp}(W \otimes \mathbf{C})$ module via the splitting j. Clearly the projection $p : S^*(W'') \to \mathscr{W}_\lambda$ induces an isomorphism onto $\mathscr{W}_\lambda / \mathscr{J}$ and we obtain an action of $\mathscr{W}_\lambda$ and $\mathfrak{sp}(W \otimes \mathbf{C})$ on $S^*(W'')$ which we will identify henceforth with $S^*(W')^*$, using $\langle\ ,\ \rangle$ as follows. We let z_j, for $1 \leqslant j \leqslant n$, be the linear functional given by

$$z_j(w') = \langle w', w_j'' \rangle.$$

We now identify $S^{*}(W')^{*}$ with $\mathscr{P}(\mathbf{C}^n)$, the algebra of polynomials in $z_1, z_2, \ldots, z_n$. We let $\frac{\partial}{\partial z_j}$ denote the derivation of the polynomials in $z_1, z_2, \ldots, z_n$ determined by $\frac{\partial}{\partial z_j}(z_k) = \delta_{jk}$. We now compute the above action, to be denoted ρ_λ, of $\mathscr{W}_\lambda$ on $\mathscr{P}(\mathbf{C}^n)$. Since $\mathscr{W}_\lambda$ is generated by W it suffices to determine how W acts on $\mathscr{P}(\mathbf{C}^n)$. The following lemma is immediate.

Lemma **6.1.**

(i) $\rho_\lambda(w_j'') = z_j$.

(ii) $\rho_\lambda(w_j') = 2i\lambda \frac{\partial}{\partial z_j}$.

We next make explicit the action of $\mathfrak{sp}(W \otimes \mathbf{C})$ which we denote by ω_λ. We observe that

$$\omega_\lambda = \rho_\lambda \circ j : \mathfrak{sp}(W \otimes \mathbf{C}) \to \operatorname{End} \mathscr{P}(\mathbf{C}^n).$$

Lemma **6.2.**

(i) $\omega_\lambda(w_j' \circ w_k') = 4\lambda \frac{\partial^2}{\partial z_j \, \partial z_k}$.

(ii) $\omega_\lambda(w_j'' \circ w_k'') = -\frac{1}{\lambda} z_j z_k$.

(iii) $\omega_\lambda(w_j' \circ w_k'') = -i\left[z_k \frac{\partial}{\partial z_j} + \frac{\partial}{\partial z_j} z_k\right]$.

Proof. — We prove (i), the proofs of (ii) and (iii) being similar:

$$\begin{aligned}
\omega_\lambda(w_j' \circ w_k') &= \rho_\lambda(j(w_j' \circ w_k')) \\
&= -\frac{1}{2\lambda} \rho_\lambda(p(w_j' \circ w_k')) \\
&= -\frac{1}{2\lambda} \rho_\lambda(p(w_j')\, p(w_k') + p(w_k')\, p(w_j')) \\
&= -\frac{1}{2\lambda} [\rho_\lambda(p(w_j'))\, \rho_\lambda(p(\omega_k')) + \rho_\lambda(p(w_k'))\, \rho_\lambda(p(\omega_j'))] \\
&= 4\lambda \frac{\partial^2}{\partial z_j \, \partial z_k}. \quad \blacksquare
\end{aligned}$$

In what follows we will let ω denote the action of $\mathfrak{sp}(W \otimes \mathbf{C})$ which we obtain by specializing λ to $2\pi i$ in the above formulas. We will call ω the ***infinitesimal Fock model*** of the oscillator representation of $\mathfrak{sp}(W \otimes \mathbf{C})$.

Lemma **6.2** *(bis).*

(i) $\omega(w'_j \circ w'_k) = 8\pi i \dfrac{\partial^2}{\partial z_j \, \partial z_k}$.

(ii) $\omega(w''_j \circ w''_k) = -\dfrac{1}{2\pi i} z_j \, z_k$.

(iii) $\omega(w'_j \circ w''_k) = -i \left[z_k \dfrac{\partial}{\partial z_j} + \dfrac{\partial}{\partial z_j} z_k \right]$.

In what follows we will need an intertwining operator from the Schrödinger model to the Fock model. On the Schwartz space $\mathscr{S}(\mathbf{R}^n)$ the operators

$$\omega(e_j) = \frac{\partial}{\partial x_j}$$
$$\omega(f_j) = 2\pi i x_j$$

generate an action of the Heisenberg algebra with central character $\lambda = 2\pi i$, whence an action σ of $\mathscr{W}_\lambda$ on $\mathscr{S}(\mathbf{R}^n)$ with $\lambda = 2\pi i$. The induced action of W' on $\mathscr{S}(\mathbf{R}^n)$ is determined by

$$\sigma(w'_j) = \frac{\partial}{\partial x_j} + 2\pi x_j \quad \text{for } 1 \leqslant j \leqslant n.$$

Thus the Gaussian φ_0 is annihilated by W' and there is a unique $\mathscr{W}_\lambda$-intertwining operator $\iota : \mathscr{S}(\mathbf{R}^n)^{(\mathfrak{u}(n))} \to \mathscr{P}(\mathbf{C}^n)$ satisfying $\iota(\varphi_0) = 1$. Here the superscript $(\mathfrak{u}(n))$ denotes the $U(n)$-finite vectors, i.e. the space of Hermite functions. The following lemma is immediate.

Lemma **6.3.**

(i) $\iota \left(\dfrac{\partial}{\partial x_j} - 2\pi x_j \right) \iota^{-1} = z_j$.

(ii) $\iota \left(\dfrac{\partial}{\partial x_j} + 2\pi x_j \right) \iota^{-1} = -4\pi \dfrac{\partial}{\partial z_j}$.

Proof. — The first formula follows from

$$\iota\sigma(w''_j) \, \iota^{-1} = \rho(w''_j)$$

and the second from

$$\iota\sigma(w'_j) \, \iota^{-1} = \rho(w'_j). \quad \blacksquare$$

7. The Fock Model for the Dual Pair O(V) × Sp(W)

We begin by recalling the basic fact that if V is a real vector space equipped with a non-degenerate symmetric bilinear form (,) and W is a real vector space equipped with a non-degenerate skew-symmetric form $\langle \; , \; \rangle$ then $(\; , \;) \otimes \langle \; , \; \rangle$ is a non-degenerate skew-symmetric form on $V \otimes W$. We wish to construct the Fock model for $(\omega, \mathfrak{sp}(V \otimes W))$ and compute how $\mathfrak{o}(V)$ and $\mathfrak{sp}(W)$ operate in this model. A possible choice of almost complex structure on $V \otimes W$ would be $\mathrm{id} \otimes J$ where J is a positive definite complex structure on W. However this almost complex structure is not positive definite. In order to obtain a positive definite almost complex structure we choose a Cartan involution θ for $\mathrm{O(V)}$ and define

$$J_0 = \theta \otimes J.$$

We will assume that we have chosen a basis $\{v_1, \ldots, v_m\}$ for V as in § 5. We choose $\theta = I_{p,q}$ and $J_0 = I_{p,q} \otimes J$. Here $I_{p,q}$ denotes the diagonal matrix with first p diagonal entries equal to $+1$ and last q diagonal entries equal to -1.

We now compute the $\pm i$-eigenspaces $(V \otimes W)'$ and $(V \otimes W)''$ of J_0 operating on $V \otimes W \otimes \mathbf{C}$. The following lemma is immediate from the definitions.

Lemma **7.1.**

(i) $(V \otimes W)' = V_+ \otimes W' + V_- \otimes W''$.

(ii) $(V \otimes W)'' = V_- \otimes W' + V_+ \otimes W''$.

Here V_+ (resp. V_-) is the span of $\{v_\alpha : 1 \leqslant \alpha \leqslant p\}$ (resp. $\{v_\mu : p + 1 \leqslant \mu \leqslant m\}$) and W' (resp. W'') is the $+i$ (resp. $-i$)-eigenspace of J operating on $W \otimes \mathbf{C}$.

Thus the underlying vector space of the Fock model for $\mathfrak{sp}(V \otimes W)$ is $S(V_- \otimes W' + V_+ \otimes W'')$. We introduce linear functionals

$$\{z_{\alpha j}, z_{\mu j} : 1 \leqslant \alpha \leqslant p, p + 1 \leqslant \mu \leqslant m, 1 \leqslant j \leqslant n\}$$

on $V_+ \otimes W' + V_- \otimes W''$ by the formulas

$$z_{\alpha j}(v \otimes w) = \langle v \otimes w, v_\alpha \otimes w_j'' \rangle, \quad 1 \leqslant \alpha \leqslant p, \quad 1 \leqslant j \leqslant n,$$

$$z_{\mu j}(v \otimes w) = \langle v \otimes w, v_\mu \otimes w_j' \rangle, \quad p + 1 \leqslant \mu \leqslant p + q, \quad 1 \leqslant j \leqslant n.$$

We use these formulas to identify the space $S(V_+ \otimes W'' + V_- \otimes W')$ with the space of polynomial functions on $V_+ \otimes W' + V_- \otimes W''$ and then with the space of polynomials in mn complex variables

$$\{z_{\alpha j}, z_{\mu j} : 1 \leqslant \alpha \leqslant p, \ p + 1 \leqslant \mu \leqslant p + q, \ 1 \leqslant j \leqslant n\}$$

to obtain an action of $\mathscr{W}_\lambda$ on $\mathscr{P}(\mathbf{C}^{mn})$ which we again denote ρ_λ. We first compute the induced action of $W \otimes \mathbf{C}$. The following lemma is immediate from the definitions.

Lemma **7.2.**

(i) $\rho_\lambda(v_\alpha \otimes w_j') = 2i\lambda \dfrac{\partial}{\partial z_{\alpha j}}$.

(ii) $\rho_\lambda(v_\alpha \otimes w_j'') = z_{\alpha j}$.

(iii) $\rho_\lambda(v_\mu \otimes w_j') = z_{\mu j}$.

(iv) $\rho_\lambda(v_\mu \otimes w_j'') = 2i\lambda \dfrac{\partial}{\partial z_{\mu j}}$.

We now compute the actions of $\mathfrak{o}(V)$ and $\mathfrak{sp}(W)$ induced by the natural inclusions $j_1 : \mathfrak{o}(V) \to \mathfrak{sp}(V \otimes W)$ and $j_2 : \mathfrak{sp}(W) \to \mathfrak{sp}(V \otimes W)$ given by

$$j_1(x) = x \otimes 1 \quad \text{and} \quad j_2(y) = 1 \otimes y.$$

We identify $\mathfrak{o}(V)$ with $\Lambda^2(V)$ using $(\ ,\ )$ and $\mathfrak{sp}(W)$ with $S^2(W)$ using $\langle\ ,\ \rangle$ and compute instead the induced maps $j_1 : \Lambda^2(V) \to S^2(V \otimes W)$ and $j_2 : S^2(W) \to S^2(V \otimes W)$ given by

$$j_1(x) = x \otimes \langle\ ,\ \rangle^* \quad \text{and} \quad j_2(y) = (\ ,\ )^* \otimes y,$$

where $\langle\ ,\ \rangle^*$ and $(\ ,\ )^*$ are the induced bilinear forms on W^* and V^* respectively. The following lemma is then immediate.

Lemma **7.3.**

(i) $j_1(v_1 \wedge v_2) = \dfrac{1}{2i}\{\sum_{j=1}^{n} (v_1 \otimes w_j') \circ (v_2 \otimes w_j'') - \sum_{j=1}^{n} (v_1 \otimes w_j'') \circ (v_2 \otimes w_j')\}.$

(ii) $j_2(w_1 \circ w_2) = \sum_{\alpha=1}^{n} (v_\alpha \otimes w_1) \circ (v_\alpha \otimes w_2) - \sum_{\mu=p+1}^{p+q} (v_\mu \otimes w_1) \circ (v_\mu \otimes w_2).$

We can now compute the actions of $\mathfrak{o}(V)$ and $\mathfrak{sp}(W)$ in the infinitesimal Fock model (with parameter λ) for $\mathfrak{sp}(V \otimes W)$. We recall that the elements $X_{\alpha\beta}$ and $X_{\mu\nu}$ in $\mathfrak{k}$ and $X_{\alpha\mu} \in \mathfrak{p}$ for $1 \leqslant \alpha, \beta \leqslant p$ and $p+1 \leqslant \mu, \nu \leqslant p+q$ are defined as follows.

Definition. — The elements $X_{\alpha\beta}$, $X_{\mu\nu}$ and $X_{\alpha\mu}$ are the elements in $\mathfrak{g}$ corresponding to $v_\alpha \wedge v_\beta$, $v_\mu \wedge v_\nu$ and $v_\alpha \wedge v_\mu$ under the isomorphism ρ between $\mathfrak{o}(V)$ and $\Lambda^2(V)$ induced by the form $(\ ,\ )$ described in § 5.

The following theorem is an immediate consequence of the previous formulas.

Theorem **7.1.** — *a) The elements* $X_{\alpha\beta}$, $X_{\mu\nu}$ *and* $X_{\alpha\mu}$ *of* $\mathfrak{o}(V)$ *operate on* $\mathscr{P}(\mathbf{C}^{mn})$ *according to the formulas*

(i) $$\omega_\lambda(X_{\alpha\beta}) = \sum_{j=1}^{n}\left(z_{\alpha j}\frac{\partial}{\partial z_{\beta j}} - z_{\beta j}\frac{\partial}{\partial z_{\alpha j}}\right),$$

(ii) $$\omega_\lambda(X_{\mu\nu}) = -\sum_{j=1}^{n}\left(z_{\mu j}\frac{\partial}{\partial z_{\nu j}} - z_{\nu j}\frac{\partial}{\partial z_{\mu j}}\right),$$

(iii) $$\omega_\lambda(X_{\alpha\mu}) = \frac{1}{2i\lambda}\sum_{j=1}^{n} z_{\alpha j} z_{\mu j} - (2i\lambda)\sum_{j=1}^{n}\frac{\partial^2}{\partial z_{\alpha j}\,\partial z_{\mu j}}.$$

b) The elements $w_j' \circ w_k'$, $w_j'' \circ w_k''$ *and* $w_j' \circ w_k''$ *of* $\mathfrak{sp}(W)$ *operate on* $\mathscr{P}(\mathbf{C}^{mn})$ *according to the formulas*

(i) $$\omega_\lambda(w_j' \circ w_k') = 4\lambda\sum_{\alpha=1}^{p}\frac{\partial^2}{\partial z_{\alpha j}\,\partial z_{\alpha k}} + \frac{1}{\lambda}\sum_{\mu=p+1}^{p+q} z_{\mu j} z_{\mu k},$$

(ii) $$\omega_\lambda(w_j'' \circ w_k'') = -\frac{1}{\lambda}\sum_{\alpha=1}^{p} z_{\alpha j} z_{\alpha k} - 4\lambda\sum_{\mu=p+1}^{p+q}\frac{\partial^2}{\partial z_{\mu j}\,\partial z_{\mu k}},$$

(iii) $$\omega_\lambda(w_j' \circ w_k'') = -i\left\{\sum_{\alpha=1}^{p}\left(z_{\alpha k}\frac{\partial}{\partial z_{\alpha j}} + \frac{\partial}{\partial z_{\alpha j}} z_{\alpha k}\right) - \sum_{\mu=p+1}^{p+q}\left(z_{\mu j}\frac{\partial}{\partial z_{\mu k}} + \frac{\partial}{\partial z_{\mu k}} z_{\mu j}\right)\right\}.$$

Corollary. — *In the case* $\lambda = 2\pi i$ *we have*

(i) $$\omega(X_{\alpha\mu}) = 4\pi \sum_{j=1}^{n} \frac{\partial^2}{\partial z_{\alpha j}\, \partial z_{\mu j}} - \frac{1}{4\pi} \sum_{j=1}^{n} z_{\alpha j}\, z_{\mu j},$$

(ii) $$\omega(w_j' \circ \omega_k') = 2i \left[4\pi \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z_{\alpha j}\, \partial z_{\alpha k}} - \frac{1}{4\pi} \sum_{\mu=p+1}^{p+q} z_{\mu j}\, z_{\mu j} \right].$$

We will need the formula for the differential d of the Lie algebra complex $(\Lambda \mathfrak{p}^* \otimes \mathscr{P}(\mathbf{C}^{mn}))^{\mathrm{K}}$. The following lemma follows immediately from the formula of § 3:

$$d = \sum_{\alpha, \mu} A_{\alpha\mu} \otimes \omega(X_{\alpha\mu}).$$

Lemma **7.4.**

$$d = 4\pi \sum_{\alpha, \mu} A_{\alpha\mu} \otimes \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_{j\alpha}\, \partial z_{j\mu}} \right) - \frac{1}{4\pi} \sum_{\alpha, \mu} A_{\alpha\mu} \otimes \left(\sum_{j=1}^{n} z_{j\alpha}\, z_{j\mu} \right).$$

Remark. — We will use the notation

$$d^+ = -\frac{1}{4\pi} \sum_{\alpha, \mu} A_{\alpha\mu} \otimes \left(\sum_{j=1}^{n} z_{j\alpha}\, z_{j\mu} \right),$$

$$d^- = +\, 4\pi \sum_{\alpha, \mu} A_{\alpha\mu} \otimes \left(\sum_{j=1}^{n} \frac{\partial^2}{\partial z_{j\alpha}\, \partial z_{j\mu}} \right),$$

whence

$$d = d^+ + d^-$$

We now specialize to the case in which $\dim W = 2$ (and $\lambda = 2\pi i$). It is shown in Lion-Vergne [15], page 183, that the usual complex structure on $\mathfrak{h} = SL_2(\mathbf{R})/SO_2$ corresponds to $D(-J)$ where J is defined on a symplectic basis $\{e, f\}$ for W by

$$Je = f \quad \text{and} \quad Jf = -e.$$

Thus the anti-holomorphic tangent space to $\mathfrak{h}$ at i is naturally identified with the $-2i$-eigenspace of $D(-J)$ (whence the $+2i$-eigenspaces of DJ) on $S^2(W \otimes \mathbf{C})$ and is consequently spanned by $\frac{1}{2i} w_1' \circ w_1'$. We obtain the following lemma. Since the index j of the previous formulas is always 1 we will drop it henceforth.

Lemma **7.5.** — *A cohomology class* $[\varphi] \in H^{\bullet}(\mathfrak{g}, \mathfrak{k}; \mathscr{P}(\mathbf{C}^m))$ *is holomorphic if and only if*

$$\left(4\pi \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z_\alpha^2} - \frac{1}{4\pi} \sum_{\mu=p+1}^{p+q} z_\mu^2 \right) [\varphi] = 0.$$

Remark. — We will abuse notation and use $\bar{\partial}$ to denote the operator $4\pi \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z_\alpha^2} - \frac{1}{4\pi} \sum_{\mu=p+1}^{p+q} z_\mu^2$ henceforth.

We will also need the formula for the class corresponding to φ_q^+ in the relative Lie algebra cohomology of $O(p, q)$ with coefficients in the Fock model. This latter class is represented by $(1 \otimes \iota)(\varphi_q^+)$ where ι is the intertwining operator from the Schrödinger model to the Fock model of the previous section.

Lemma **7.6.**

$$(1 \otimes \iota)(\varphi_q^+) = \left(-\frac{1}{4\pi}\right)^q \sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_q} \omega_{\alpha_1 p+1} \wedge \ldots \wedge \omega_{\alpha_q p+q}$$

Proof. — The lemma is immediate from the defining formula for φ_q^+ of § 5 and the formulas

$$\iota(\varphi_0) = 1 \quad \text{and} \quad \iota\left(x_j - \frac{1}{2\pi}\frac{\partial}{\partial x_j}\right)\iota^{-1} = -\frac{1}{2\pi} z_j. \quad \blacksquare$$

We conclude this section with a simplifying change of variable. We replace z_α and z_μ by $\sqrt{4\pi}\, z_\alpha$ and $\sqrt{4\pi}\, z_\mu$. Then $\bar{\partial}$, d^+, d^- and $(1 \otimes \iota)(\varphi_q^+)$ are transformed to (respectively)

$$\bar{\partial}' = \sum_{\alpha=1}^{p} \frac{\partial^2}{\partial z_\alpha^2} - \sum_{\mu=p+1}^{p+q} z_\mu^2,$$

$$(d^+)' = -\sum_{\alpha, \mu} A_{\alpha\mu} \otimes z_\alpha z_\mu,$$

$$(d^-)' = \sum_{\alpha, \mu} A_{\alpha\mu} \otimes \frac{\partial^2}{\partial z_\alpha \partial z_\mu},$$

$$(1 \otimes \iota)(\varphi_q^+)' = (-1)^q (4\pi)^{-q/2} \sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_q} \omega_{\alpha_1 p+1} \wedge \ldots \wedge \omega_{\alpha_q p+q}.$$

We drop the prime superscripts henceforth letting $\bar{\partial}$, d^+, d^- denote the right-hand sides of the above formulas. By applying the inverse change of variable to ψ_{q-1} below we obtain the following lemma:

Lemma **7.7.** — *The class* $[\varphi_q^+]$ *is holomorphic if there exists*

$$\psi_{q-1} \in [\Lambda^{q-1} \mathfrak{p}^* \otimes \mathscr{P}(\mathbf{C}^m)]^K$$

such that

$$\bar{\partial}\Big(\sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_q} \omega_{\alpha_1 p+1} \wedge \ldots \wedge \omega_{\alpha_q p+q}\Big) = d\psi_{q-1}.$$

We find an explicit formula for ψ_{q-1} in the next section. We will change our notation there and use φ_q^+ to denote the form $\sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_q} \omega_{\alpha_1 p+1} \wedge \ldots \wedge \omega_{\alpha_q p+q}$.

8. The proof that φ_q^+ is a holomorphic class

We define a " homotopy operator " $h : [\Lambda^k \mathfrak{p}^* \otimes \mathscr{P}(\mathbf{C}^m)]^K \to [\Lambda^{k-1} \mathfrak{p}^* \otimes \mathscr{P}(\mathbf{C}^m)]^K$ by the formula

$$h = \sum_{\alpha, \mu} A_{\alpha\mu}^* \otimes z_\mu \frac{\partial}{\partial z_\alpha}.$$

Here $A_{\alpha\mu}^*$ denotes interior multiplication by $e_{\alpha\mu}$ where $\{ e_{\alpha\mu} \}$ is the basis of $\mathfrak{p}$ dual to the basis $\{ \omega_{\alpha\mu} \}$ of $\mathfrak{p}^*$. We now prove the following " homotopy formula ".

Lemma **8.1.**

$$dh + hd = 1 \otimes E_- \Delta_+ - 1 \otimes \rho_- E_+ - p 1 \otimes \rho_- + \sum_{\alpha,\beta} D_{\beta\alpha} \otimes \frac{\partial^2}{\partial z_\alpha \partial z_\beta} + \sum_{\mu,\nu} D_{\nu\mu} \otimes z_\mu z_\nu.$$

Remarks and Notation. — We have used the following notation in the statement of the lemma

$$E_- = \sum_\mu z_\mu \frac{\partial}{\partial z_\mu}, \quad E_+ = \sum_\alpha z_\alpha \frac{\partial}{\partial z_\alpha}, \quad \rho_- = \sum_\mu z_\mu^2, \quad \text{and} \quad \Delta_+ = \sum_\alpha \frac{\partial^2}{\partial z_\alpha^2}.$$

The operators $D_{\beta\alpha}$ and $D_{\nu\mu}$ are the derivations of the exterior algebra on $\mathfrak{p}^*$ determined by

$$D_{\beta\alpha}\, \omega_{\gamma\mu} = \delta_{\alpha\gamma}\, \omega_{\beta\mu} \quad \text{and} \quad D_{\nu\mu}\, \omega_{\alpha\lambda} = \delta_{\lambda\mu}\, \omega_{\alpha\nu}.$$

We observe that $D_{\beta\alpha}$ and $D_{\nu\mu}$ may be interpreted in terms of the natural action σ of $\mathfrak{gl}_p(V_+)$ and $\mathfrak{gl}_q(V_-)$ on $\mathfrak{p}^* = V_+^* \otimes V_-$ as follows. Let $e_{ij} \in \mathfrak{gl}_n(\mathbf{R})$ denote the matrix with 1 in the i, j position and zero elsewhere. Then

$$D_{\beta\alpha} = \sigma(e_{\alpha\beta}) \quad \text{and} \quad D_{\nu\mu} = \sigma(e_{\mu\nu}).$$

Proof of Lemma **8.1.** — We begin by computing $d^+ h + hd^+$. We have

$$\begin{aligned} d^+ h &= -\Big(\sum_{\beta,\nu} A_{\beta\nu} \otimes z_\beta z_\nu\Big) \circ \left(\sum_{\alpha,\mu} A^*_{\alpha\mu} \otimes z_\mu \frac{\partial}{\partial z_\alpha}\right) \\ &= -\sum_{\alpha,\beta,\mu,\nu} A_{\beta\nu} A^*_{\alpha\mu} \otimes z_\beta z_\mu z_\nu \frac{\partial}{\partial z_\alpha}, \\ hd^+ &= -\left(\sum_{\alpha,\mu} A^*_{\alpha\mu} \otimes z_\mu \frac{\partial}{\partial z_\alpha}\right) \circ \Big(\sum_{\beta,\nu} A_{\beta\nu} \otimes z_\beta z_\nu\Big) \\ &= -\sum_{\alpha,\beta,\mu,\nu} A^*_{\alpha\mu} A_{\beta\nu} \otimes z_\beta z_\mu z_\nu \frac{\partial}{\partial z_\alpha} - \sum_{\alpha,\mu,\nu} A^*_{\alpha\mu} A_{\alpha\nu} \otimes z_\mu z_\nu. \end{aligned}$$

Hence

$$d^+ h + hd^+ = -\sum_{\alpha,\beta,\mu,\nu} \{A_{\beta\nu}, A^*_{\alpha\mu}\} \otimes z_\beta z_\mu z_\nu \frac{\partial}{\partial z_\alpha} - \sum_{\mu,\nu} \Big(\sum_\alpha A^*_{\alpha\mu} A_{\alpha\nu}\Big) \otimes z_\mu z_\nu.$$

Here $\{A, B\}$ denotes the anticommutator $AB + BA$. The Clifford identities [7], p. 112, imply

$$\{A_{\beta\nu}, A^*_{\alpha\mu}\} = \delta_{\alpha\beta}\, \delta_{\mu\nu}$$

and we obtain

$$d^+ h + hd^+ = -1 \otimes \Big(\sum_\mu z_\mu^2\Big) \left(\sum_\alpha z_\alpha \frac{\partial}{\partial z_\alpha}\right) - \sum_{\mu,\nu} \Big(\sum_\alpha A^*_{\alpha\mu} A_{\alpha\nu}\Big) \otimes z_\mu z_\nu.$$

We rewrite the last term as

$$\sum_\mu \Big(\sum_\alpha A^*_{\alpha\mu} A_{\alpha\mu}\Big) \otimes z_\mu^2 + \sum_{\mu \neq \nu} \Big(\sum_\alpha A^*_{\alpha\mu} A_{\alpha\nu}\Big) \otimes z_\mu z_\nu.$$

Applying the Clifford identities again,

$$A^*_{\alpha\mu} A_{\alpha\mu} = I - A_{\alpha\mu} A^*_{\alpha\mu},$$

$$A^*_{\alpha\mu} A_{\alpha\nu} = - A_{\alpha\nu} A^*_{\alpha\nu} \quad \text{for } \mu \neq \nu,$$

we obtain

$$\sum_{\mu,\nu} (\sum_\alpha A^*_{\alpha\mu} A_{\alpha\nu}) \otimes z_\mu z_\nu = p1 \otimes \sum_\mu z_\mu^2 - \sum_{\mu,\nu} (\sum_\alpha A_{\alpha\nu} A^*_{\alpha\mu}) \otimes z_\mu z_\nu.$$

Thus

$$d^+ h + hd^+ = -1 \otimes (\sum_\mu z_\mu^2) \left(\sum_{\alpha=1}^{p} z_\alpha \frac{\partial}{\partial z_\alpha}\right) - p1 \otimes \sum_\mu z_\mu^2 + (\sum_\alpha A_{\alpha\nu} A^*_{\alpha\mu}) \otimes (\sum_{\mu,\nu} z_\mu z_\nu).$$

We now leave to the reader the task of verifying that for any quadruple α, β, μ, ν the composition $A_{\alpha\nu} A^*_{\alpha\mu}$ is a derivation of the exterior algebra on $\mathfrak{p}^*$. Hence $\Sigma_\alpha A_{\alpha\nu} A^*_{\alpha\mu}$ is determined by its action on $\mathfrak{p}^*$. It is then immediate that

$$D_{\nu\mu} = \sum_\alpha A_{\alpha\nu} A^*_{\alpha\mu}.$$

We obtain

$$dh^+ + hd^+ = -1 \otimes \rho_- E_+ - p1 \otimes \rho_- + \sum_{\mu,\nu} D_{\nu\mu} \otimes z_\mu z_\nu.$$

We next compute the corresponding formula for $d^- h + hd^-$. We first compute

$$\begin{aligned} d^- h &= \left(\sum_{\beta,\nu} A_{\beta\nu} \otimes \frac{\partial^2}{\partial z_\beta \partial z_\nu}\right) \circ \left(\sum_{\alpha,\mu} A^*_{\alpha\mu} \otimes z_\mu \frac{\partial}{\partial z_\alpha}\right) \\ &= \sum_{\alpha,\beta,\mu,\nu} A_{\beta\nu} A^*_{\alpha\mu} \otimes z_\mu \frac{\partial^3}{\partial z_\alpha \partial z_\beta \partial z_\nu} + \sum_{\alpha,\beta,\mu} A_{\beta\mu} A^*_{\alpha\mu} \otimes \frac{\partial^2}{\partial z_\alpha \partial z_\beta}. \end{aligned}$$

Also

$$hd^- = \sum_{\alpha,\beta,\mu,\nu} A^*_{\alpha\mu} A_{\beta\nu} \otimes z_\mu \frac{\partial^3}{\partial z_\alpha \partial z_\beta \partial z_\nu},$$

whence

$$d^- h + hd^- = \sum_{\alpha,\beta,\mu,\nu} \{A_{\beta\nu}, A^*_{\alpha\mu}\} \otimes z_\mu \frac{\partial^3}{\partial z_\alpha \partial z_\beta \partial z_\nu} + \sum_{\alpha,\beta,\mu} A_{\beta\mu} A^*_{\alpha\mu} \otimes \frac{\partial^2}{\partial z_\alpha \partial z_\beta}.$$

We use the Clifford identities again to obtain

$$\sum_{\alpha,\beta,\mu,\nu} \{A_{\beta\nu}, A^*_{\alpha\mu}\} \otimes z_\mu \frac{\partial^3}{\partial z_\alpha \partial z_\beta \partial z_\nu} = 1 \otimes \left(\sum_\mu z_\mu \frac{\partial}{\partial z_\mu}\right)\left(\sum_\alpha \frac{\partial^2}{\partial z_\alpha}\right) = 1 \otimes E_- \Delta_+.$$

Hence

$$d^- h + hd^- = 1 \otimes E_- \Delta_+ + \sum_{\alpha,\beta} (\sum_\mu A_{\beta\mu} A^*_{\alpha\mu}) \otimes \frac{\partial^2}{\partial z_\alpha \partial z_\beta}.$$

We have as before

$$D_{\beta\alpha} = \sum_{\mu} A_{\beta\mu} A^*_{\alpha\mu},$$

and we obtain

$$d^- h + hd^- = 1 \otimes E_- \Delta_+ + \sum_{\alpha,\beta} D_{\alpha\beta} \otimes \frac{\partial^2}{\partial z_\alpha \partial z_\beta}.$$

Combining the formulas for $d^+ h + hd^+$ and $d^- h + hd^-$, we obtain the lemma. ■

Before proving that φ_q^+ is a holomorphic class we need an elementary lemma giving the behavior of φ_q^+ under $\mathfrak{gl}_q(\mathbf{R})$ and $\mathfrak{gl}_p(\mathbf{R})$.

Lemma **8.2.**

(i) $(D_{\nu\mu} \otimes 1)\, \varphi_q^+ = \delta_{\nu\mu}\, \varphi_q^+$.

(ii) $(D_{\beta\alpha} \otimes 1)\, \varphi_q^+ = \left(1 \otimes z_\alpha \dfrac{\partial}{\partial z_\beta}\right) \varphi_q^+$.

Proof. — The first formula follows immediately from the formula (assuming $\mu < \nu$)

$$D_{\nu\mu}(\ldots \wedge \omega_{\alpha\mu} \ldots \wedge \omega_{\beta\nu} \ldots) = \ldots \wedge \omega_{\alpha\nu} \ldots \wedge \omega_{\beta\nu} \ldots$$

and so if $\mu \neq \nu$ the terms in $D_{\nu\mu}\, \varphi_q^+$ cancel (in pairs if $\alpha \neq \beta$). The second formula is more subtle.

$$\begin{aligned}
z_\alpha \frac{\partial}{\partial z_\beta} \varphi_q^+ &= z_\alpha \frac{\partial}{\partial z_\beta} \Big(\sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_q}\, \omega_{\alpha_1\, p+1} \ldots \omega_{\alpha_q\, p+q} \Big) \\
&= \sum_{i=1}^{q} \sum_{\alpha_1, \ldots, \alpha_q} z_\alpha z_{\alpha_1}, \ldots \frac{\partial}{\partial z_\beta}(z_{\alpha_i}) \ldots z_{\alpha_q}\, \omega_{\alpha_1\, p+1} \ldots \omega_{\alpha_q\, p+q} \\
&= \sum_{i=1}^{q} \sum_{\alpha_1, \ldots, \alpha_q} z_\alpha z_{\alpha_1} \ldots \delta_{\alpha_i \beta} \ldots z_{\alpha_q}\, \omega_{\alpha_1\, p+1} \ldots \omega_{\alpha_i\, p+i} \ldots \omega_{\alpha_q\, p+q} \\
&= \sum_{i=1}^{q} \sum_{\alpha_1, \ldots, \alpha_{q-1}} z_{\alpha_1} \ldots z_{\alpha_{q-1}} z_\alpha\, \omega_{\alpha_1\, p+1} \ldots \omega_{\beta p+i} \ldots \omega_{\alpha_{q-1}\, p+q}, \\
D_{\beta\alpha}\, \varphi_q^+ &= \sum_i \sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_q}\, \omega_{\alpha_1\, p+1} \ldots D_{\beta\alpha}(\omega_{\alpha_i\, p+i}) \ldots \omega_{\alpha_q\, p+q} \\
&= \sum_i \sum_{\alpha_1, \ldots, \alpha_q} z_{\alpha_1} \ldots z_{\alpha_i} \ldots z_{\alpha_q}\, \delta_{\alpha\alpha_i}\, \omega_{\alpha_1\, p+1} \ldots \omega_{\beta p+i} \ldots \omega_{\alpha_q\, p+q} \\
&= \sum_i \sum_{\alpha_1, \ldots, \alpha_{q-1}} z_{\alpha_1} \ldots z_{\alpha_{q-1}} z_\alpha\, \omega_{\alpha_1\, p+1} \ldots \omega_{\beta p+i} \ldots \omega_{\alpha_{q-1}\, p+q}. \quad ■
\end{aligned}$$

We can now find ψ_{q-1}.

Lemma **8.3.**

$$d(h\varphi_q^+) = (p + q - 1)\, \omega(w_1' \circ w_1')\, \varphi_q^+.$$

Proof. — Since φ_q^+ is closed, we have

$$\begin{aligned} dh\varphi_q^+ &= (dh + hd)\, \varphi_q^+ \\ &= (-1 \otimes \rho_- \mathrm{E}_+)\, \varphi_q^+ - (p1 \otimes \rho_-)\, \varphi_q^+ \\ &\qquad + \left(\sum_{\alpha,\beta} \mathrm{D}_{\beta\alpha} \otimes \frac{\partial^2}{\partial z_\alpha\, \partial z_\beta}\right) \varphi_q^+ + \left(\sum_{\mu,\nu} \mathrm{D}_{\nu\mu} \otimes z_\mu\, z_\nu\right) \varphi_q^+. \end{aligned}$$

Here we have made the observation that

$$(1 \otimes \mathrm{E}_-\, \Delta_+)\, \varphi_q^+ = 0.$$

Now we apply the previous lemma to deduce

$$\left(\sum_{\mu,\nu} \mathrm{D}_{\nu\mu} \otimes z_\mu\, z_\nu\right) \varphi_q^+ = \rho_-\, \varphi_q^+$$

and

$$\begin{aligned} \left(\sum_{\alpha,\beta} \mathrm{D}_{\beta\alpha} \otimes \frac{\partial^2}{\partial z_\alpha\, \partial z_\beta}\right) \varphi_q^+ &= \sum_{\alpha,\beta} \left(1 \otimes \frac{\partial^2}{\partial z_\alpha\, \partial z_\beta}\right) \circ (\mathrm{D}_{\beta\alpha} \otimes 1)\, \varphi_q^+ \\ &= \sum_{\alpha,\beta} \left(1 \otimes \frac{\partial^2}{\partial z_\alpha\, \partial z_\beta}\right) \circ \left(1 \otimes z_\alpha \frac{\partial}{\partial z_\beta}\right) \varphi_q. \end{aligned}$$

An easy calculation establishes the formula

$$\sum_{\alpha,\beta} \frac{\partial^2}{\partial z_\alpha\, \partial z_\beta} \circ z_\alpha \frac{\partial}{\partial z_\beta} = (p+1)\, \Delta_+ + \mathrm{E}_+\, \Delta_+.$$

We obtain

$$\left(\sum_{\alpha,\beta} \mathrm{D}_{\beta\alpha} \otimes \frac{\partial^2}{\partial z_\alpha\, \partial z_\beta}\right) \varphi_q^+ = (p+1)\, \Delta_+\, \varphi_q^+ + \mathrm{E}_+\, \Delta_+\, \varphi_q^+.$$

But

$$\mathrm{E}_+\, \Delta_+\, \varphi_q^+ = (q-2)\, \Delta_+\, \varphi_q^+.$$

Therefore

$$\begin{aligned} dh\varphi_q^+ &= -\, q\rho_-\, \varphi_q^+ - p\rho_-\, \varphi_q^+ + \rho_-\, \varphi_q^+ + (p+q-1)\, \Delta_+\, \varphi_q^+ \\ &= (p+q-1)\, [\Delta_+\, \varphi_q^+ - \rho_-\, \varphi_q^+) \\ &= (p+q-1)\, \omega(w_1' \circ w_1')\, \varphi_q^+. \quad \blacksquare \end{aligned}$$

Thus if we define $\psi_{q-1} = \dfrac{1}{p+q-1}\, h\varphi_q^+$, then

$$\bar\partial \varphi_q^+ = d\psi_{q-1}$$

and we have proved that $[\varphi_q^+]$ is a holomorphic class.

9. The calculation of the positive semi-definite Fourier coefficients of $\theta_\varphi(\eta)$

We first review the results obtained in our earlier papers [13], [14], for the positive definite Fourier coefficients a_β of $\theta_\varphi(\eta)$ for φ defined as at the end of § 5. For these φ, the forms $c_i(\varphi)$ come from invariant forms on D, not just $\mathrm{D}_{\mathrm{U}_i}$. For the definition of the lattice $\mathscr{L}$, see Theorem 2 of the introduction. We also remind the reader that the notation $G = \Pi_{\alpha=1}^{r} G^{(\alpha)}$ was introduced in § 2.

Theorem **9.1.** — *Assume* $G^{(1)} = O(p, q)$ *and* β *is positive definite. Then* $a_\beta = 0$ *unless* $2\beta \in \mathscr{L}$. *In this case*

$$a_\beta = 2^{-nq/2} e^{-\pi \Sigma_{\alpha=1}^{r} \operatorname{tr} \beta^{(\alpha)}} \int_{C_{2\beta}} \eta.$$

Proof. — This theorem is proved for the case in which M is compact in [13] and extended to the case in which M has finite volume in [14]. ■

In fact there is an exceptional case not covered by [14], namely the case in which $q = 1$ and $n = p - 1$. In this case C_U is an infinite geodesic. This case may be treated directly by the reader. An analogous result for much more general η is proved in [18]. For the definition of the lattice $\mathscr{L}$, see Theorem 3 of the introduction.

Theorem **9.2.** — *Assume* $G^{(1)} = U(p, q)$, *then* $a_\beta = 0$ *unless* $2\beta \in \mathscr{L}$. *In this case*

$$a_\beta = i^{-nq} e^{-\pi \Sigma_{\alpha=1}^{r} \operatorname{tr} \beta^{(\alpha)}} \int_{C_{2\beta}} \eta.$$

Proof. — This theorem is proved for the case which M is compact in [13]. The extension to the case in which M is non-compact of finite volume follows immediately from Theorem 2.1 of [14] together with the explicit results of [13]; that is, the estimate in the appendix and the formula $\kappa = i^{-nq}$ of Theorem 6.4. ■

We now define the Euler form e_q on $SO_0(p, q)/SO(p) \times SO(q)$ and the q-th Chern form c_q on $SU(p, q)/S(U(p) \times U(q))$.

The Euler form e_q is zero if q is odd and is given for $q = 2\ell$ by the formula

$$e_q = \left(-\frac{1}{4\pi}\right)^{\ell} \frac{1}{\ell!} \sum_{\sigma \in S_q} \operatorname{sgn}(\sigma)\, \Omega_{\sigma(1), \sigma(2)} \ldots \Omega_{\sigma(2\ell-1), \sigma(2\ell)}$$

where

$$\Omega_{\mu\nu} = \sum_{\alpha=1}^{p} \omega_{\alpha\mu}\, \omega_{\alpha\nu}.$$

The q-th Chern form c_q is given by

$$c_q = \left(\frac{-i}{2\pi}\right)^{q} \frac{1}{q!} \sum_{\sigma, \bar\sigma \in S_q \times S_q} \operatorname{sgn}(\sigma\bar\sigma)\, \Omega_{\sigma(1), \bar\sigma(1)} \ldots \Omega_{\sigma(q), \bar\sigma(q)}.$$

In other words we alternate separately over holomorphic and anti-holomorphic indices. Here the curvature $\Omega_{\mu\nu}$ is given by

$$\Omega_{\mu\nu} = \sum_{\alpha=1}^{p} \bar\xi_{\alpha\nu} \wedge \xi_{\alpha\nu},$$

and $\{\xi_{\alpha\mu}\}$ is the basis for $(\mathfrak{p}^*)^+$ chosen in § 5.

We can now state our formulas for the positive semi-definite Fourier coefficients. Once again it is easy to see that $a_\beta = 0$ unless $2\beta \in \mathscr{L}$.

Theorem **9.3.** — *In case* $G^{(1)} = O(p, q)$ *and rank* $\beta = t$ *with* $t < n$ *and* $2\beta \in \mathscr{L}$, *we have*

$$a_\beta(\theta_\varphi(\eta)) = 2^{-nq/2}\, e^{-\pi \Sigma_{\alpha=1}^{r} \operatorname{tr} \beta^{(\alpha)}} \int_{C_{2\beta}} \eta \wedge e_q^{n-t}.$$

Proof. — By the Corollary to Theorem 4.1 we have

$$a_\beta(\theta_\varphi(\eta)) = \int_M \eta \wedge \theta(Z; \beta, t).$$

We choose a set of Γ-orbit representatives $\mathscr{C}_\beta = \{X_1, \ldots, X_\ell\}$ in $\mathscr{Q}_{\beta,t}$ and unfold the integral to obtain

$$a_\beta(\theta_\varphi(\eta)) = \sum_{i=1}^{\ell} \int_{\Gamma_{X_i}\backslash D} \eta \wedge \varphi(Z, X_i).$$

Thus the Theorem will follow if we can prove that for $X \in \mathscr{Q}_{\beta,t}$ we have

$$\int_{\Gamma_X\backslash D} \eta \wedge \varphi(Z, X) = 2^{-nq/2}\, e^{-\pi \Sigma_{\alpha=1}^{r} \operatorname{tr} \beta^{(\alpha)}} \int_{C_U} \eta \wedge e_q^{n-t}.$$

Both sides are multiplicative for coverings so it suffices to prove the above formula for the case that Γ is neat. In this case Γ_U, the stabilizer of U, acts trivially on U since $V = U + U^\perp$ is a rational splitting and $(\ ,\ )\,|\,U$ is positive definite. Hence $\Gamma_X = \Gamma_U$ and we may rewrite the left-hand side of the above equation as

$$F_\varphi(X) = \int_{\Gamma_U\backslash D} \eta \wedge \varphi(Z, X).$$

We may now allow X to be irrational in U^n since $\varphi(Z, X)$ is well-defined for all elements of U^n and the domain of integration does not depend on X, only on span X. We will need the following three lemmas to complete the proof of the theorem.

Lemma **9.1.** — *Let* $\sigma \in SO(n)^r$. *Then* $F_\varphi(X\sigma) = F_\varphi(X)$.

Proof. — The domain of the integral does not change and neither does the integrand because

$$\varphi(Z, X\sigma) = \varphi(Z, X)$$

for $\sigma \in SO(n)^r$. ■

Lemma **9.2.** — *Let* $b = (x_{i_1}, \ldots, x_{i_t})$ *be the basis for* $U^{(1)}$ *obtained from* $X^{(1)}$ *by refining* $X^{(1)}$ *to a basis starting on the left (see* § 2*). Then there exists* $\sigma_1 \in SO_n(\mathbf{R})$ *such that* $Y = X^{(1)}\sigma_1$ *is of the form*

$$Y = (y_{i_1}, \ldots, y_{i_t}, 0, \ldots, 0),$$

where $(y_{i_1}, \ldots, y_{i_t})$ *is a basis for* $U^{(1)}$ *in the same orientation class as* b.

Proof. — Let $g_1 \in \mathrm{GL}_n(\mathbf{R})^r$ satisfy $\mathrm{X}^{(1)} g_1 = (\mathrm{X}^{(1)})'$ with

$$(\mathrm{X}^{(1)})' = (x_{i_1}, \ldots, x_{i_t}, 0, \ldots, 0).$$

Let P be the parabolic subgroup of $\mathrm{GL}_n(\mathbf{R})$ which stabilizes the subspace formed by the first t standard basis vectors and $\mathrm{Q} = {}^t\mathrm{P}$. Then $q \in \mathrm{Q}$ implies that $(\mathrm{X}^{(1)})' q^{-1}$ is of the form $(x'_{i_1}, \ldots, x'_{i_t}, 0, \ldots, 0)$, where $(x'_{i_1}, \ldots, x'_{i_t})$ is a basis for $\mathrm{U}^{(1)}$ in the same orientation class as $(x_{i_1}, \ldots, x_{i_t})$. Now decompose g_1 according to the decomposition $\mathrm{GL}_n(\mathbf{R}) = \mathrm{O}_n(\mathbf{R}).\mathrm{Q}$. We have $g_1 = \sigma_1 q$ and $\mathrm{X}^{(1)} \sigma_1$ is of the form

$$\mathrm{X}^{(1)} \sigma_1 = (y_{i_1}, \ldots, y_{i_t}, 0, \ldots, 0),$$

where $(y_{i_1}, \ldots, y_{i_t})$ is a basis for U in the same orientation class as $(x_{i_1}, \ldots, x_{i_t})$. If, in fact, $\sigma_1 \in \mathrm{SO}_n(\mathbf{R})$, we are done. Otherwise we replace σ_1 by $\sigma_1 \iota_1$ where ι_1 is the element of $\mathrm{O}_n(\mathbf{R})$ that changes the sign of the $(t+1)$st coordinate and leaves all other coordinates fixed. ∎

We now define an element $\mathrm{X}' \in \mathrm{V}^n = \bigoplus_{\alpha=1}^r (\mathrm{V}^{(\alpha)})^n$ by

$$(\mathrm{X}')^{(1)} = \mathrm{Y}, \qquad (\mathrm{X}')^{(i)} = \mathrm{X}^{(i)} \quad \text{for } i = 1, 2, \ldots, r.$$

We observe that $\sigma = (\sigma_1, 1, \ldots, 1) \in \mathrm{SO}_n(\mathbf{R})^r$ satisfies $\mathrm{X}\sigma = \mathrm{X}'$. Hence we have (by Theorem 5.2 (i))

$$\varphi(\mathrm{Z}, \mathrm{X}) = \varphi(\mathrm{Z}, \mathrm{X}'),$$

and by Lemma 9.1 we have

$$\mathrm{F}_\varphi(\mathrm{X}) = \mathrm{F}_\varphi(\mathrm{X}').$$

Remark. — The formula $\varphi(\mathrm{Z}, \mathrm{X}) = \varphi(\mathrm{Z}, \mathrm{X}')$ implies that if $\dim \operatorname{span} \mathrm{X}^{(1)} < n$, then $\varphi(\mathrm{Z}, \mathrm{X})$ is invariant under $\mathrm{O}^n(\mathbf{R})^r$. Indeed, if $\iota \in \mathrm{O}_n(\mathbf{R})^r$ is given by $\iota = (\iota_1, 1, \ldots, 1)$ we have

$$\varphi(\mathrm{Z}, \mathrm{X}\sigma\iota) = \varphi(\mathrm{Z}, \mathrm{X}'\,\iota) = \varphi(\mathrm{Z}, \mathrm{X}') = \varphi(\mathrm{Z}, \mathrm{X}).$$

Since $\varphi(\mathrm{Z}, \mathrm{X})$ is known to be invariant under $\mathrm{SO}_n(\mathbf{R}) \times \mathrm{O}_n(\mathbf{R})^{r-1}$ the claim follows.

Lemma **9.3.**

$$\varphi^+_{nq}(\mathrm{Z}, \mathrm{X}') = 2^{-(n-t)q/2}\, \varphi^+_{tq}(\mathrm{Z}, (y_{i_1}, \ldots, y_{i_t})) \wedge e_q^{n-t}.$$

Proof. — We have by definition

$$\varphi^+_{nq}(\mathrm{Z}, (x_1, \ldots, x_n)) = \varphi^+_q(\mathrm{Z}, x_1) \wedge \ldots \wedge \varphi^+_q(\mathrm{Z}, x_n),$$

and the lemma follows from the formula [12], Proposition 5.1, which implies

$$\varphi^+_q(0) = 2^{-q/2}\, e_q. \quad ∎$$

We can now prove the theorem. Indeed

$$\mathrm{F}_\varphi(\mathrm{X}') = \left(\int_{\Gamma_{\mathrm{U}}\backslash \mathrm{D}} (\eta \wedge e_q^{n-t} \wedge \varphi^+_{tq}(\mathrm{Z}, (y_{i_1}, \ldots, y_{i_t})))\right) \left(\prod_{\alpha=2}^{r} e^{-\pi \operatorname{tr}(\beta^{(\alpha)})}\right).$$

Now $Y' = (y_{i_1}, \ldots, y_{i_t})$ is a non-singular frame and $\Gamma_U \backslash D$ is the associated tube. Note that if $\beta' = (Y', Y')$, then $\operatorname{tr} \beta^{(1)} = \operatorname{tr}(\beta')$. The theorem follows from the usual " Thom Lemma ", i.e. Theorem 2.1 of [14] applied with η replaced by $\eta \wedge e_q^{n-t}$.

Theorem **9.4.** — *In case* $G^{(1)} = U(p, q)$ *and rank* $\beta = t$ *with* $t < n$ *and* $2\beta \in \mathscr{L}$, *we have*

$$a_\beta(\theta_\varphi(\eta)) = i^{-nq} e^{-\pi \Sigma_{\alpha=1} \operatorname{tr} \beta^{(\alpha)}} \int_{C_{2\beta}} \eta \wedge c_q^{n-t}.$$

Proof. — The proof of this theorem is analogous to that of the preceding one using Proposition 5.2 of [12], implying that

$$\varphi^+_{q,q}(0) = i^{-q} c_q. \quad \blacksquare$$

Appendix. The estimation of $||\theta_\varphi(Z; \beta, R)||$

We will use the notation of § 4. We let $\varphi \in (\Lambda^i \mathfrak{p}^* \otimes \mathbf{S}(V^n))^{K_0}$, β and R be as in § 4 and define an i-form $\theta_\varphi(Z; \beta, R)$ on $E = \Gamma_R \backslash D$ by

$$\theta_\varphi(Z; \beta, R) = \sum_{X \in \mathscr{Q}_{\beta, R}} \varphi(Z, X).$$

We consider the following series depending on $g \in G$, a lattice $\Lambda \subset V^n$ and a parameter $\lambda \in \mathbf{R}_+$

$$S(g; \Lambda, \beta, R, \lambda) = \sum_{X \in \mathscr{Q}_{\beta, R}} \exp(-\pi\lambda \operatorname{tr}(g^{-1} X, g^{-1} X)_0).$$

Here $(\ ,\ )_0$ denotes the positive definite form on V constructed in § 4. Clearly there exists a constant C such that

$$||\theta(gZ_0, \beta, R)|| \leqslant CS(g; L^n, \beta, R, \lambda)$$

for a suitable λ (depending on v). We choose a suitable λ_0 once and for all and define

$$S(g; \Lambda, \beta, R) = \sum_{X \in \mathscr{Q}_{\beta, R}} \exp(-\pi\lambda_0 \operatorname{tr}(g^{-1} X, g^{-1} X)_0).$$

We redefine $(\ ,\ )_0$ so that $\lambda_0 = 1$. Hence to prove Lemma 4.3 it is sufficient to prove the following theorem. Let Ω be a fundamental domain for $\Gamma \cap N$ in N.

Theorem **A.** — *Let* $T > 0$ *be given. Then there exist positive numbers* C *and* ε *such that for all* $t \in \mathbf{R}$ *with* $t \leqslant T$ *and all* $n \in \Omega$,

$$(*) \qquad S(na(t)\, mk; \Lambda, \beta, R) \leqslant Ce^{-\varepsilon e^{-2t}}.$$

Proof. — We first claim that if the inequality $(*)$ holds for some lattice $\Lambda \subset V^n$ and some $\varepsilon \in \mathbf{R}_+$ then it holds for any other lattice Λ in V^n and another (possibly different) $\varepsilon \in \mathbf{R}_+$. First if $(*)$ holds for a lattice Λ then it holds for any sublattice of Λ since all the terms on the left-hand side are positive. Second if $(*)$ holds for Λ and

$\mu \in \mathbf{R}$ then $(*)$ holds for the dilated lattice $\mu\Lambda$ provided we change the constant ε on the right-hand side to $\mu^2 \varepsilon$. But if Λ is a fixed lattice and L is any other lattice, there exists $N \in \mathbf{N}$ such that $L \subset \frac{1}{N}\Lambda$. The claim is now established.

We now replace L (if necessary) by the lattice $L_0 + L_1 + L_2$, where

$$L_0 = L \cap R, \qquad L_1 = L \cap W, \qquad L_2 = L \cap R',$$

adapted to the rational Witt decomposition $V = R + W + R'$ of § 4. We may also assume that

$$L_1 = L_1' + L_1'' + L_1''',$$

where

$$L_1' = L_1 \cap U_1, \qquad L_2'' = L_1 \cap U_2, \qquad L_1''' = L_1 \cap U_3$$

and $W = U_1 + U_2 + U_3$ is the decomposition of § 4. We will now prove $(*)$ for the lattice $\Lambda = L^n$. We write $X = X^0 + X'$, where $X^0 \in L_0^n$ and $X' \in (L_1')^n$. The condition $X \in \mathcal{Q}_{\beta, R}$ imposes only a congruence condition on X^0, but $X' \in (L_1')^n$ must satisfy $(X', X') = \beta$ as well as the congruence condition. We can majorize S (up to a constant multiple) by allowing X^0 to range freely over L_0^n provided X^0 spans R and allowing X' to range freely over $(L_1')^n$ provided $(X', X') = \beta$. This we accomplish by first passing to bigger lattices $\tilde{L}_0^n$ and $(\tilde{L}_1')^n$ which contain the projection of the vectors in h then returning to sums over L_0^n and $(L_1')^n$ by the arguments above. We will henceforth drop the prime on L_1' so L_1 is now a lattice in U_1. We obtain

$$S(g; \Lambda, \beta, R) = \sum_{X \in L_0^n}{}' \exp(-\pi \operatorname{tr}(g^{-1}X, g^{-1}X)_0) \sum_{Y \in L_1^n \cap \mathcal{Q}_\beta} \exp(-\pi \operatorname{tr}(g^{-1}Y, g^{-1}Y)_0).$$

Here the notation $\Sigma'_{X \in L_0^n}$ means that we sum only over those elements of L_0^n that span R, and $\mathcal{Q}_\beta$ now means

$$\mathcal{Q}_\beta = \{ Y \in U_1^n : (Y, Y) = \beta \}.$$

We now write $g = m_1 m_2 a(t) n$ with $m_1 \in SL(R)$, $m_2 \in O(W)$ (or $U(W)$ in case $G = U(p, q)$), $a(t) \in A$, $n \in N$. Here we have identified GL(R) and O(W) with subgroups of G in the usual way using the Witt decomposition of V constructed in § 4. We have

$$S(m_1 m_2 a(t) n; \Lambda, \beta, R) = \Big(\sum_{X \in L_0}{}' \exp(-\pi e^{-2t} \operatorname{tr}(m_1^{-1}X, m_1^{-1}X)_0)\Big) \cdot \Big(\sum_{Y \in L_1^n \cap \mathcal{Q}_\beta} \exp(-\pi(n^{-1}m_2^{-1}Y, n^{-1}m_2^{-1}Y)_0)\Big).$$

We begin by estimating the first sum. We claim that there exists a positive constant ε, independent of $m_1 \in M$ and $X \in L_0^n$, such that

$$\operatorname{tr}(m_1^{-1}X, m_1^{-1}X)_0 \geqslant \varepsilon.$$

To see this, note that the quadratic forms $(m_1^{-1})^*(\ ,\ )_0$, $m_1 \in M_1$, all have the same discriminant relative the lattice L (the determinant of the matrix of the form relative a $\mathbf{Z}$-basis for L). The claim then follows from the next elementary lemma.

Lemma **A-1.** — *Let* $L \subset \mathbf{R}^k$ *be a lattice and* $\{X_1, \ldots, X_n\} \subset L$ *be a spanning set for* $\mathbf{R}^k$. *Let* $(\ ,\ )$ *be a symmetric positive definite form on* $\mathbf{R}^k$ *with discriminant* d_0 *relative the lattice* L. *Then*

$$\sum_{i=1}^{n} (X_i, X_i) \geqslant k d_0^{1/k}.$$

Proof. — Suppose $\{X_{j1}, \ldots, X_{jk}\} \subset \{X_1, \ldots, X_n\}$ is a basis for $\mathbf{R}^k$. Since $\sum_{i=1}^{n} (X_i, X_i) \geqslant \sum_{i=1}^{k}(X_{ji}, X_{ji})$, it suffices to prove the inequality for the case $n = k$; that is, the case in which $\{X_1, \ldots, X_n\}$ is a basis for $\mathbf{R}^k$. Let S be the matrix of $(\ ,\ )$ relative $\{X_1, \ldots, X_k\}$. Let R be a rotation such that RSR^{-1} is a diagonal matrix with diagonal entries $d_1, \ldots, d_k$. Then $d_1, d_2, \ldots, d_k$ are positive and their product d satisfies $d \geqslant d_0$. Now, by a well-known inequality,

$$(\det S)^{1/k} = d^{1/k} = (d_1 d_2 \ldots d_k)^{1/k} \leqslant \frac{1}{k}(d_1 + d_2 + \ldots + dr) \leqslant \frac{1}{k} \operatorname{tr} S.$$

Therefore

$$\operatorname{tr} S \geqslant k d^{1/k} \geqslant k d_0^{1/k}. \quad \blacksquare$$

Thus we obtain the claim with $\varepsilon = \frac{1}{2} r d^{1/r}$ (recall $\dim R = r$). We now write

$$\begin{aligned} \operatorname{tr}(m_1^{-1} X, m_1^{-1} X)_0 &= \frac{1}{2} \operatorname{tr}(m_1^{-1} X, m_1^{-1} X)_0 + \frac{1}{2} \operatorname{tr}(m_1^{-1} X, m_1^{-1} X)_0 \\ &\geqslant \varepsilon + \frac{1}{2} \operatorname{tr}(m_1^{-1} X, m_1^{-1} X)_0. \end{aligned}$$

We obtain

$$\begin{aligned} S(m_1 m_2 a(t)\, n; \Lambda, \beta, R) \leqslant \exp(-\varepsilon\pi e^{-2t}) \\ \cdot \Big(\sum_{X \in L_0^n}{}' \exp\Big(-\frac{1}{2}\pi e^{-2t} \operatorname{tr}(m_1^{-1} X, m_2^{-1} X)_0\Big)\Big) \\ \Big(\sum_{Y \in L_1^n \cap \mathcal{D}_\beta} \exp(-\pi \operatorname{tr}(n^{-1} m_2^{-1} Y, n^{-1} m^{-1} Y)_0)\Big). \end{aligned}$$

We bound separately each of the two functions

$$F_1(m_1) = \sum_{X \in L_0^n}{}' \exp\Big(-\frac{1}{2}\pi \operatorname{tr}(m_1^{-1} X, m_2^{-1} X)_0\Big)$$

and

$$F_2(nm_2) = \sum_{Y \in L_1^n \cap \mathcal{D}_\beta} \exp(-\pi \operatorname{tr}(n^{-1} m_2^{-1} Y, n^{-1} m_2^{-1} Y)_0).$$

We will bound F_1 and F_2 for the case $G = SO(p, q)$. The case $G = SU(p, q)$ is identical (up to changing $\mathbf{R}$ to $\mathbf{C}$ and O(W) to U(W)).

We first estimate F_1 as a function on SL(R). We may assume that $r \geqslant 2$, for if $r = 1$, then SL(R) is the trivial group. For the rest of this argument we will use g (instead of m_1) to denote a variable element in $SL_r(\mathbf{R})$. Also since the elements $X \in V^n$ are in fact in R^n we may forget about the ambient group $O(p, q)$ and ambient vector space V. Thus the summation runs over elements X contained in a certain lattice $\Lambda \subset (R)^n$ with the property that span X = R. We now choose a basis for R and identify R with $\mathbf{R}^r$, R^n with the space $M(r, n; \mathbf{R})$ of r by n matrices with real entries and SL(R) with $SL_r(\mathbf{R})$. We will prove that the sum

$$F(g; \Lambda, (\ ,\ )) = \sum_{X \in \Lambda}{}' \exp(-\pi \operatorname{tr}(g^{-1} X, g^{-1} X))$$

is a bounded function on $SL_r(\mathbf{R})$ for any lattice $\Lambda \subset M(r, n; \mathbf{R}) \cong (\mathbf{R}^r)^n$ and any positive definite form $(\ ,\ )$ on $\mathbf{R}^r$. Here the superscript prime indicates we are summing over matrices in Λ of rank r. As before it suffices to prove boundedness for a single lattice in $(\mathbf{R}^r)^n$ and a single positive definite form $(\ ,\ )$ on $\mathbf{R}^r$. We choose $\Lambda = M(r, n; \mathbf{Z}) \cong L^n$ with $L = \mathbf{Z}^r \subset \mathbf{R}^r$ the standard integral lattice and take $(\ ,\ )$ to be the sum of squares of the coordinates. We put $\Gamma = SL_r(\mathbf{Z})$.

Since F is clearly left-invariant under Γ, it suffices to prove that F is bounded on a fundamental domain $\mathscr{D}$ for Γ in $SL_r(\mathbf{R})$. We recall the definition of the Siegel set $\mathfrak{S}_{t,\omega} \subset SL_r(\mathbf{R})$ for $t \in \mathbf{R}_+$ and ω the fundamental domain for $\Gamma \cap N$ in N, where N denotes the subgroup of strictly upper triangular matrices of $SL_r(\mathbf{R})$. We have

$$\mathfrak{S}_{t,\omega} = \left\{ g \in SL_r(\mathbf{R}) : g = na(d)\,k \text{ with } n \in \omega \text{ and } \frac{d_i}{d_{i+1}} \geqslant t \text{ for } 1 \leqslant i \leqslant r-1 \right\}.$$

Here and for the rest of this section we will use the following notation. The letter d will denote an element $(d_1, d_2, \ldots, d_r)$ of $(\mathbf{R}_+)^r$ and $a(d)$ will denote the diagonal matrix with diagonal entries $(d_1, d_2, \ldots, d_r)$. We let A denote the subgroup of diagonal matrices and $A_t \subset A$ be the subset

$$A_t = \left\{ a(d) \in A : \alpha_i(a(d)) = \frac{d_i}{d_{i+1}} \geqslant t \text{ for } 1 \leqslant i \leqslant r-1 \right\}.$$

By the basic theorem of reduction theory there exists a compact subset C of $SL_r(\mathbf{R})$ such that

$$\mathscr{D} = \mathfrak{S}_{t,\omega} \cup C$$

for t sufficiently small. Thus it suffices to bound F on $\mathfrak{S}_{t,\omega}$. Now if $g \in \mathfrak{S}_{t,\omega}$ we may write g as $g = nak$ with $n \in \omega$, $a \in A_t$ and $k \in SO(r)$. We obtain

$$F(g) = F(nak) = \sum_{X \in L^n} \exp(-\pi \operatorname{tr}(a^{-1} n^{-1} X, a^{-1} n^{-1} X)).$$

Since A acts on N by sums of the simple roots $\alpha_1, \alpha_2, \ldots, \alpha_r$ with $\alpha_i(d) = \dfrac{d_i}{d_{i+1}} \geqslant t$, it is clear that the set $\bigcup_{a \in A_t} a^{-1} \omega^{-1} a$ is a relatively compact subset of N. Now we have

$$\operatorname{tr}(a^{-1} n^{-1} X, a^{-1} n^{-1} X) = \operatorname{tr}((a^{-1} n^{-1} a)\, a^{-1} X, (a^{-1} n^{-1} a)\, a^{-1} X).$$

Clearly then we may find a constant c such that for all $g \in \mathfrak{S}_{t,\omega}$ we have

$$\operatorname{tr}(g^{-1}X, g^{-1}X) \geqslant c \operatorname{tr}(a^{-1}X, a^{-1}X),$$

where $g = nak$ is the Iwasawa decomposition of g. It remains to estimate

$$F(a) = \sum_{X \in L^n}{}' \exp(-\pi \operatorname{tr}(a^{-1}X, a^{-1}X)), \quad a \in A_t.$$

It is convenient to parametrize A_t somewhat differently. We put $c = (c_1, c_2, \ldots, c_r)$ and let $\widetilde{a}(c)$ be the diagonal matrix with diagonal entries $(c_1^{-1}, c_2^{-1}, \ldots, c_r^{-1})$. Thus $\widetilde{a}(c) \in A_t$ if and only if $\frac{c_{i+1}}{c_i} \geqslant t$ for $1 \leqslant i \leqslant r-1$. We record the elementary consequences:

(a) $\quad \dfrac{c_i}{c_1} \geqslant t^{i-1}$ for $1 \leqslant i \leqslant r$,

(b) $\quad \dfrac{c_i}{c_r} \leqslant t^{i-r}$ for $1 \leqslant i \leqslant r$,

(c) $\quad c_r \geqslant t^{(r-2)/2}\, c_1^{-1/(r-1)}$.

We have identified X with an r by n integral matrix (x_{ij}) of rank r by writing

$$X_j = \sum_{i=1}^{r} x_{ij}\, e_i \quad \text{for } 1 \leqslant j \leqslant n.$$

Here $\{e_1, e_2, \ldots, e_r\}$ is the standard basis for $\mathbf{R}^r$. We obtain

$$\operatorname{tr}(X, X) = \sum_{i,j} x_{ij}^2$$

and

$$\operatorname{tr}(\widetilde{a}(c)^{-1}X, \widetilde{a}(c)^{-1}X) = \sum_{i=1}^{r} c_i^2 \left(\sum_{j=1}^{n} x_{ij}^2\right) = \sum_{i=1}^{r} c_i^2 \,\|X^i\|^2.$$

Here we let X^i denote the vector in $\mathbf{R}^n$ given by the i-th row of the matrix (x_{ij}) and $\|X^i\|^2$ denotes the sum of the squares of the entries in the i-th row. Thus we have passed from $(X_1, \ldots, X_n)$, an n-frame in $\mathbf{R}^r$ to $(X^1, \ldots, X^r)$, an r-frame in $\mathbf{R}^n$. We note that, since (x_{ij}) has rank r, the vectors $X^1, \ldots, X^r$ are independent. In particular none of them is zero. We now change the meaning of L and let L denote the integral lattice in $\mathbf{R}^n$.

We can show that $F(\widetilde{a}(c))$ is bounded on A_t. We break up A_t into two regions. one where c_1 is large, say $c_1 \geqslant 1$, and the second where c_1 is small, say $c_1 < 1$. The estimation of the sum in the first region is easy. By (a) above we have

$$\sum_{i=1}^{r} c_i^2 \,\|X^i\|^2 \geqslant c_1^2 \sum_{i=0}^{r-1} t^{2i} \,\|X^i\|^2 \geqslant \alpha \sum_{i=0}^{r-1} \|X^i\|^2,$$

where α is a positive constant depending on t ($\alpha = 1$ if $t \geqslant 1$ or $\alpha = t^{2r-2}$ if $t < 1$).

We obtain

$$F(\widetilde{a}(c)) \leqslant \sum_{X \in L^r} \exp(-\alpha\pi \operatorname{tr}(X, X)) \leqslant C.$$

We now prove boundedness in the second region. We have (with Σ' indicating a sum over r-tuples of *independent* lattice vectors)

$$\begin{aligned}
\mathrm{F}(\tilde{a}(c)) &= \sum_{\mathrm{L}^r}{}' \exp(-\pi \sum_{i=1}^{r} c_i^2 \,\|\, \mathrm{X}^i \,\|^2) \\
&\leqslant \sum_{(\mathrm{L}-\{0\})^r} \exp(-\pi \sum_{i=1}^{r} c_i^2 \,\|\, \mathrm{X}^i \,\|^2) \\
&\leqslant \prod_{i=1}^{r} \Big(\sum_{\mathrm{X}\in \mathrm{L}-\{0\}} \exp(-\pi c_i^2 \,\|\, \mathrm{X} \,\|^2)\Big).
\end{aligned}$$

We again use (a) with $c_i \geqslant t^{i-1} c_1$ for $i = 1, 2, \ldots, r-1$ in the first $r-1$ products above to obtain

$$\mathrm{F}(\tilde{a}(c)) \leqslant \prod_{i=1}^{r-1} \Big(\sum_{\mathrm{L}-\{0\}} \exp(\pi c_1^2 t^{2i-2} \,\|\, \mathrm{X} \,\|^2)\Big) \Big(\sum_{\mathrm{L}-\{0\}} \exp(-\pi c_r^2 \,\|\, \mathrm{X} \,\|^2)\Big).$$

We now use (c) to obtain

$$\mathrm{F}(\tilde{a}(c)) \leqslant \prod_{i=1}^{r-1} \Big(\sum_{\mathrm{L}-\{0\}} \exp(-\pi c_1^2 t^{2i-2} \,\|\, \mathrm{X} \,\|^2)\Big) \Big(\sum_{\mathrm{L}-\{0\}} \exp(-\pi t^{r-2} c_1^{-2/(r-1)} \,\|\, \mathrm{X} \,\|^2)\Big).$$

We recall the behavior of the sum $\sum_{\mathrm{X}\in\mathrm{L}} \exp(-u \,\|\, \mathrm{X} \,\|^2)$ for u tending to zero. The following lemma is an immediate consequence of Poisson summation.

Lemma **A-2.** — *There exists a constant* C_1 *such that*

$$\sum_{\mathrm{X}\in\mathrm{L}} \exp(-u \,\|\, \mathrm{X} \,\|^2) \sim \mathrm{C}_1 u^{-n/2} \quad \textit{as } u \to 0.$$

Corollary. — *Given any* $\mathrm{T} \in \mathbf{R}$ *there exists a constant* C_2 *such that*

$$\sum_{\mathrm{X}\in\mathrm{L}} \exp(-u \,\|\, \mathrm{X} \,\|^2) \leqslant \mathrm{C}_2 u^{-n/2} \quad \textit{for } 0 \leqslant u \leqslant \mathrm{T}.$$

We also have the following elementary estimate:

Lemma **A-3.** — *There exist positive constants* α *and* C_3 *such that*

$$\sum_{\mathrm{X}\in\mathrm{L}-\{0\}} \exp(-v \,\|\, \mathrm{X} \,\|^2) \leqslant \mathrm{C}_3 e^{-\alpha v} \quad \textit{for } v \geqslant 1.$$

Applying Lemma A-2 to the first factor of the product on the right-hand side of the previous estimate for $\mathrm{F}(\tilde{a}(c))$ and Lemma A-3 to the second factor we obtain positive constants $\mathrm{C}(t)$ and $\alpha(t)$ such that in region 2 we have

$$\mathrm{F}(\tilde{a}(c)) \leqslant \mathrm{C}(t)\, c_1^{-n(r-1)} \exp(-\alpha(t)\, c_1^{-2/(r-1)}).$$

Clearly this formula implies that $\mathrm{F}(\tilde{a}(c))$ is bounded in region 2.

We now estimate $\mathrm{F}_2(nm_2)$. The set $\mathrm{S} = \{(n^{-1})^* (\ ,\ ) : n \in \Omega\}$ is a compact set of positive-definite forms. Hence there exists $\varepsilon > 0$ such that for all $n \in \Omega$

$$(n^{-1})^* (\ ,\ ) \geqslant \varepsilon(\ ,\ ).$$

We see then that it suffices to prove that

$$F_2(m_2) = \sum_{Y \in L_1^n \cap \mathscr{Q}_\beta} \exp(-\pi(m_2^{-1} Y, m_2^{-1} Y)_0)$$

is bounded on O(W). In fact we have the following lemma.

Lemma **A-4.**

$$F_2(m_2) \leqslant F_2(1).$$

Proof. — We first recall the Berger decomposition of O(W) associated to the involution $\sigma \in O(W)$ with -1-eigenspace equal to U_1 and $+1$-eigenspace equal to $U_2 + U_3$ (the splitting $W = U_1 + U_2 + U_3$ was defined in § 4). We let H be the centralizer of σ. Let $\ell = \min\{\dim U_1, \dim U_3\}$. We choose orthogonal sets $\{u_1, \ldots, u_\ell\} \subset U_1$ and $\{v_1, v_2, \ldots, v_\ell\} \subset U_3$ with $(u_i, u_i) = 1$, $1 \leqslant i \leqslant \ell$, and $(v_i, v_i) = -1$, $1 \leqslant i \leqslant \ell$. For $r = (r_1, r_2, \ldots, r_\ell) \in \mathbf{R}^\ell$, we define $b_r \in O(W)$

$$\left.\begin{aligned} b_r u_j &= ch(r_j)\, u_j + sh(r_j)\, v_j \\ b_r v_j &= sh(r_j)\, u_j + ch(r_j)\, v_j \end{aligned}\right\} \text{ for } 1 \leqslant j \leqslant \ell$$

and $b_r = 1$ on the orthogonal complement of $T = \operatorname{span}\{u_1, \ldots, u_\ell, v_1, \ldots, v_\ell\}$ for $(\ ,\ )$ in W. We observe that the orthogonal complements of T computed relative to $(\ ,\ )_0$ and $(\ ,\ )$ coincide. We define $B \subset O(W)$ by

$$B = \{b_r : r \in \mathbf{R}^\ell\}.$$

We also let $H_1 \subset H$ be the subgroup which acts by the identity on U_1. Then the Berger decomposition is given by

$$O(W) = HBK = O(U_1)\, H_1\, BK.$$

Thus any $m_2 \in M$ may be decomposed as $m_2 = hh_1 bk$ with $h \in O(U_1)$, $h_1 \in H_1$, $b \in B$ and $k \in K \cap O(W)$, whence

$$\operatorname{tr}(m_2^{-1} Y, m_2^{-1} Y)_0 = \operatorname{tr}(b^{-1} h_1^{-1} h^{-1} Y, b^{-1} h_1^{-1} h^{-1} Y)_0.$$

But in Lemma A-2 (ii) of the appendix to [13], it is proved that for all $b \in B$ and Y as above

$$\operatorname{tr}(b^{-1} Y, b^{-1} Y)_0 \geqslant \operatorname{tr}(Y, Y)_0.$$

We obtain

$$\operatorname{tr}(m_2^{-1} Y, m_2^{-1} Y)_0 \geqslant \operatorname{tr}(h^{-1} Y, h^{-1} Y)_0.$$

Thus it remains to prove that

$$F_2(h) = \sum_{Y \in L^n \cap \mathscr{Q}_\beta} \exp(-\pi(h^{-1} Y, h^{-1} Y)_0)$$

is bounded. Now in case β is positive semi-definite, then $(\ ,\ ) \mid U_1 = (\ ,\ )_0 \mid U_1$ by construction and

$$(h^{-1} Y, h^{-1} Y) = (Y, Y).$$

The sum is necessarily finite and we find that F_2 is indeed bounded.

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