Lecture 7: The Action from the Right of Invertible Matrices on Bases

The n by n Matrix $C(\mathscr{B})$ Associated to a Basis $\mathscr{B}$ for F^n

In this lecture we will assume we are working in F^n (or $\mathbb{R}^n$) (for general V we will assume V has dimension n and we have chosen a basis for V). So in what follows $V = F^n$.

Let $\mathscr{B} = (v_1, v_2, \cdots, v_m)$ be a basis for F^n. Then we define the matrix $C(\mathscr{B})$ of the basis $\mathscr{B}$ as the n by n matrix $C(\mathscr{B})$ given by

$$\begin{matrix} & v_1 & v_2 & \dots & v_m \\ C(\mathscr{B}) = \Bigg(& \downarrow & \downarrow & \dots & \downarrow & \Bigg) \end{matrix}$$

So the i^{th} column of $C(\mathscr{B})$ is the coordinates of the vector $v_i \in F^n$. I will leave the following proposition to you as an exercise.

Proposition

$C(\mathscr{B})$ is an invertible matrix. Conversely given any n by n invertible matrix D there exists a unique basis $\mathscr{B}$ of V such that $C(\mathscr{B}) = D$.

The Action from the Right of Invertible n by n Matrices on Row Vectors

In the next slide we will define the action of invertible n by n matrices on bases, so a "row vector of vectors" $\mathscr{B} = (v_1, v_2, \cdots, v_n$. This definition is motivated by the formula for the action of n by n matrices on row vectors $(x_1, x_2, \cdots, x_n)$ (of scalars) which is simply the definition of the matrix product of a 1 by n matrix and an n by n matrix. We recall the formula.

So let $u = (x_1, x_2, \cdots, x_n)$ be a row vector and $A = (a_{ij})$ be an n by n matrix. By definition of matrix multiplication the matrix product $u \cdot A$ is given by

$$u \cdot A = (\sum_{j=1}^{n} x_j a_{j1}, \sum_{j=1}^{n} x_j a_{j2}, \cdots, \sum_{j=1}^{n} x_j a_{jn}).$$

But we may interchange x_j and $a_{jk}, 1 \leq k \leq n$ in each term to obtain

$$u \cdot A = (\sum_{j=1}^{n} a_{j1} x_j, \sum_{j=1}^{n} a_{j2} x_j, \cdots, \sum_{j=1}^{n} a_{jn} x_j.$$

If you replace the numbers x_j by vectors v_j in this formula you get the formula of the next slide for $\mathscr{B} \bullet A$.

The Action from the Right of Invertible n by n Matrices on Bases

Suppose $\mathscr{B} = (v_1, v_2, \cdots, v_n)$ is a basis and $T \in L(V,\ V)$. Then we define the left action of T on $\mathscr{B}$ by

$$T\mathscr{B} = (Tv_1, Tv_2, \cdots, Tv_n)$$

Now suppose$A$ is an n by n invertible matrix. Then we define the action (**from the right**) of A on $\mathscr{B}$ by

$$\begin{aligned}\mathscr{B} \bullet A =& (v_1, v_2, \cdots, v_n) \bullet A \\ =& (\sum_{j=1}^{n} a_{j1}v_j, \sum_{j=1}^{n} a_{j2}v_j, \cdots, \sum_{j=1}^{n} a_{jn}v_j)\end{aligned}$$

Two Examples

Example 1

$$(v_1, v_2) \bullet \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = (a_{11}v_1 + a_{21}v_2, a_{12}v_1 + a_{22}v_2)$$

Example 2

Proposition

Suppose $\mathscr{B} = (b_1, b_2, \cdot, b_n)$ *is a basis for* V *and* $v \in V$ *has coordinates* $(x_1, x_2, \cdots, x_n)$. *Then*

$$(b_1, b_2, \cdots, b_n) \bullet \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} x_i b_i = v$$

Remark

So if we right-multiply a basis for V by a column vector of scalars we get a vector in V and the column vector is the coordinates of v relative to the basis.

Further Discussion of Example 2

Strictly speaking Example 2 is not an " example " because

$$A = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

is not an invertible matrix. But in fact we can right multiply a basis $\mathscr{B} = (b_1, b_2, \cdot, b_n)$ for F^n by an n by m matrix A and get an m-tuple of vectors in F^n - not a basis unless $m = n$ and A is invertible. In Example 2 we multiplied the basis $\mathscr{B}$ by the n by 1 matrix

$$A = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

and the result was a single vector in V. **This example will be the key to proving the change of basis formula for the coordinates of a vector in Lecture 8.**

A Useful Proposition

The next proposition will be very useful in computing with change of bases. It states that the right action of an n by n matrix A on a bases $\mathscr{B}$ corresponds under C to right multiplication $C(\mathscr{B})$ by A. This will make it easy to prove theorems about the action of invertible matrices on bases. So the mapping C that takes bases to matrices carries the right action of an invertible n by n matrix A on a basis $\mathscr{B}$ to the right multiplication by A on the matrix $C(\mathscr{B})$ associated to $\mathscr{B}$.

Proposition

Suppose $C(\mathscr{B}) = D$. *Then*

$$C(\mathscr{B} \bullet A) = DA.$$

Proof.

By definition of the action of A on $\mathscr{B}$ we have

$$\begin{aligned}\mathscr{B} \bullet A =&(v_1, v_2, \cdots, v_n) \bullet A \\ =&(\sum_{j=1}^{n} a_{j1} v_j, \sum_{j=1}^{n} a_{j2} v_j, \cdots, \sum_{j=1}^{n} a_{jn} v_j)\end{aligned}$$

Let $D_1, D_2, \cdots, D_m$ be the columns of D. Then the columns of the matrix DA are $\sum_{j=1}^{n} a_{j1} D_j$, $\sum_{j=1}^{n} a_{j2} D_j, \cdots$, $\sum_{j=1}^{n} a_{jn} D_j)$. Since D_j corresponds to v_j under C this proves the proposition.

□

The Proposition has an important corollary

Corollary

$\mathscr{B} \bullet A_1 = \mathscr{B} \bullet A_2 \iff A_1 = A_2$

Proof.

$\mathscr{B} \bullet A_1 = \mathscr{B} \bullet A_2 \iff C(\mathscr{B}) A_1 = C(\mathscr{B}) A_2$. Left multiply by

A New Formula for the Matrix of a Linear Transformation

Theorem

Suppose $\mathcal{B} = (b_1, b_2, \cdot, b_n)$ *is a basis for* V *and* $T \in L(V,\ V)$*. Let* $M(T) =_{\mathscr{B}} T_{\mathscr{B}}$ *be the matrix of* T *relative to* $\mathcal{B}$*. Then*

$$(T(b_1), T(b_2), \cdots, T(b_n)) = (b_1, b_2, \cdots, b_n) \bullet M(T)$$

Proof.

By definition the matrix $M(T)$ is the matrix (a_{ij}) where the entries (a_{ij}) satisfy

$$T(b_j) = \sum_i b_i a_{ij} = \sum_i a_{ij} b_i, 1 \leq j \leq n. \tag{1}$$

We now compute the right-hand side of the equation in the theorem. But also by definition (third slide)

$$(b_1, b_2, \cdots, b_n) \bullet M(T) = (\sum_i b_i a_{i1}, \sum_i b_i a_{i2}, \cdots, \sum_i b_i a_{in}$$

which is the same as the right-hand side of (1) and the Theorem follows. □

Problem

Suppose $T \in L(V W$ and $\mathcal{B} = (b_1, b_2, \cdots, b_n)$ is a basis for V and $\mathcal{C} = (c_1, c_2, \cdots, c_n)$ is a basis for W. What is the formula analogous to the formula of the previous Theorem for ${}_{\mathcal{C}}[T]_{\mathcal{B}}$?