Lecture 15: Properties of the Determinant

Last time we proved the existence and uniqueness of the determinant $\det : M_{n\times n}(F) \longrightarrow F$ satisfying 5 axioms.
For this lecture we will be using the last three axioms dealing with how $\det(A)$ behaves when elementary row operations are performed on A.

Type 1. Interchange two rows of A to get a new matrix B. Then

$$\det(B) = -\det(A).$$

Type 2. Replace the i-th row R_i of A by $R_i + cR_j$ where R_j is a different row to get B. Then

$$\det(B) = \det(A).$$

Type 3. Multiply the i-th row R_i of A by c (and don't change the other rows) to get B. Then

$$\det(A) = c\det(A).$$

The key point in what follows is that each of the three operations can be realized by multiplying A on the left by an "elementary matrix".

Proposition (1)

Let E be the result of applying one of the above elementary operations to the identity matrix and B be the result of applying the same operation to A. Then

$$B = EA.$$

Proof: I will prove it only for the case $n = 2$.

Type 1 Interchande the two rows.

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad B = \begin{pmatrix} c & d \\ a & b \end{pmatrix} \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \end{aligned}$$

$$EA = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix} = B.$$

Type 2 (Add λR_1 to R_2)

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad B = \begin{pmatrix} a & b \\ c+\lambda a & d+\lambda b \end{pmatrix} \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \end{aligned}$$

$$EA = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix} = B.$$

Type 3 (Multiply the second row by λ)

$$\begin{aligned} A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad , \quad B = \begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix} \\ I &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad E = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \end{aligned}$$

$$EA = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix} = B.$$

In what follows, I will let $\mathscr{E}(A)$ denote the result of applying an elementary row operation $\mathscr{E}$ to a matrix A and E be the elementary matrix $\mathscr{E}(I)$ so Proposition (1) can be restated as

$$\mathscr{E}(A) = \underbrace{\mathscr{E}(I)}_{E} \bullet A.$$

We know compute $\det E$ for the three types of elementary matrices.

Lemma (1)

- *Type 1*
  $\det(E) = \det(\mathscr{E}(I)) = -\det(I) = -1.$
  (The second equality hold by Axiom (iii)).
- *Type 2*
  $\det(E) = \det(\mathscr{E}(I)) = \det(I) = 1.$
  (The second equality hold by Axiom (iv)).
- *Type 3*
  $\det(E) = \det(\mathscr{E}(I)) = c\det(I) = c.$
  (The second equality hold by Axiom (v)).

We can now prove:

Lemma (2)

Let E be an elementary matrix and A be an arbitrary $n \times n$ matrix. Then

$$\det(EA) = \det(E)\det(A).$$

Proof. By proposition 1, we have

$$EA\mathscr{E}(A)$$

and so

$$\det(EA) = \det(\mathscr{E}(A)). \quad (*)$$

Now evaluate the RHS of $(*)$ by axioms (iii), (iv) and (v) to show that

$$\det(\mathscr{E}(A)) = \det(E)\det(A),$$

using Lemma 1 to evaluate $\det(E)$. For example, we saw $\det(E) = -1$. So

$$\det(E)\det(A) = (-1)\det(A) = -\det(A).$$

By axiom (iii)

$$\det(EA) = \det(\mathscr{E}(A)) = -\det(A).$$

Lemma (3)

Let E_1, …, E_k be elementary matrices and B be an arbitrary n by n matrix. Then

$$\det(E_1 \dots E_k B) = \det(E_1 \dots E_k)\det(B).$$

Proof. By induction on k. We have already proved the case of $k = 1$. Suppose it is true and consider

$$D = \det(E_1 \underbrace{E_2 \dots E_k E_{k+1} B}_{A})$$

Put $A = E_2 \dots E_k E_{k+1} B$ so the above determinant becomes $D = \det(E_1 A)$. But by Lemma (2) we have

$$\det(E_1 A) = \det(E_1)\det(A)$$

and by the induction hypothesis we have

$$\begin{aligned} D &= \det(E_1)\det(E_2 \dots E_k E_{k+1} B) \\ &= \det(E_1)\underbrace{\det(E_2 \dots E_k E_{k+1})\det(B)}_{\text{by the induction hypothesis}} \\ &= \det(E_1 E_2 \dots E_k E_{k+1} \det(B) \end{aligned}$$

The last equality is Lemma 2 with $A = E_2 \dots E_k E_{k+1}$.

Corollary

$$\det(E_1 \dots E_k) = \det(E_1) \dots \det(E_k) \neq 0.$$

Proof.

$$\begin{aligned} \det(E_1 E_2 \dots E_k) &= \det(E_1)\det(E_2 \dots E_k) \\ &= \det(E_1)\det(E_2)\dots\det(E_k). \end{aligned}$$

The last equality follows by induction. We now apply Lemma (3) to prove two important properties of determinants.

Theorem ((1))

A n by n matrix A is invertible if and only $\det(A) = 0$.

Proof. Recall that A can be reduced by elementary row operations to a matrix B which is the identity if A is invertible and had row of zeroes if A is not invertible.

So we have

$$E_1 E_2 \dots E_k A = B$$

and hence

$$\det(E_1 E_2 \dots E_k A) = \det(B).$$

By Lemma (3) we have

$$\det(E_1 E_2 \dots E_k) \det(A) = \det(B).$$

By the previous corollary we have

$$\det(E_1 \dots E_k) \neq 0,$$

so

$$\det(A) = 0 \iff \det(B) = 0.$$

But since a matrix with a row of zeroes has determinant equal to zero we see

$$\det(A) = \begin{cases} \det(I) = 1, & \text{if A is invertible} \\ 0, & \text{if A is not invertible} \end{cases}$$

□

Theorem ((2))

Let A and B by n by matrices. Then

$$\det(AB) = \det(A)\det(B).$$

Proof.

Case 1: A is invertible. Then A is a product of elementary matrices

$$A = E_1 E_2 \dots E_k$$

(See the beginning of the proof on the previous page.) Then

$$\begin{aligned} \det(AB) &= \det(E_1 \dots E_k B) \\ &= \det(E_1 \dots E_k)\det(B) \\ &= \det(A)\det(B). \end{aligned}$$

Case 2: A is not invertible, so $\det(A) = 0$. So, it suffices to show $\det(AB) = 0$. We can write $A = E_1 \dots E_k C$ where C has bottom row zero. Also $\det(E_1 \dots E_k CB) = \det(E_1 \dots E_k \det(CB)$. But since C has bottom row zero so does CB. Hence $\det(CB)$.

Corollary

$$\det A^{-1} = \frac{1}{\det(A)}.$$

Proof.

$$AA^{-1} = I$$

so

$$\det\left(AA^{-1}\right) = 1$$

so, using Theorem (2)

$$\det(A)\det\left(A^{-1}\right) = 1$$

□

Expansion by Minors

We now give two kinds of formulas for computing determinants (text, page 150).

Theorem

1. *Expansion by minors on the j-th column:*

$$\begin{aligned}\det(A) &= (-1)^{j+1}a_{1j}\det(A_{1j}) + (-1)^{j+2}a_{2j}\det(A_{2j}) + \ldots \\ &\quad +(-1)^{j+n}a_{nj}\det(A_{nj}).\end{aligned}$$

2. *Expansion by minors on the i-th row:*

$$\begin{aligned}\det(A) &= (-1)^{i+1}a_{i1}\det(A_{i1}) + (-1)^{i+2}a_{2j}\det(A_{i2}) + \ldots \\ &\quad +(-1)^{i+n}a_{in}\det(A_{in}).\end{aligned}$$

In these formulas, A_{ij} is a the matrix A with the i-th row and j-th column deleted. The terms $(-1)^{i+j}$ provide alternating signs depending on the position (i, j) in the matrix.

$$\begin{pmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ \vdots & & & \dots & \end{pmatrix}$$

We now give a formula for the inverse of a matrix A assuming $\det(A) \neq 0$ (text, page 151).
We first define the n by n matrix of signed cofactors $C = C(A) = (c_{ij})$

$$c_{ij} = (-1)^{i+j} \det(A_{ij}).$$

We then define the classical adjoint and adjugate of A denoted $\operatorname{adj}(A)$ by

$$\operatorname{adj}(A) = {}^tC = \text{ the transpose of } C$$

(See the text, page 146.)
In the text $\operatorname{adj}(A)$ is denoted A^*. This is bad notation because we will need the notation A^* later.

We then have

Theorem

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}((A)).$$

Example. The 2 by 2 case

$$\begin{aligned}
A &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \\
C(A) &= \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \\
\operatorname{adj}(A) &= {}^tC(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \\
A^{-1} &= \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}
\end{aligned}$$

The Volume Form D on a n-dimensional Vector Space

We conclude by relatin what we have done with the text, Chapter 5.

The text starts with a function

$$D : \underbrace{F^n \times F^n \times \ldots F^n}_{c \text{ copies}} \longrightarrow F$$

satisfying the axioms on page 134. The text calls D a **determinant factor**, I will call it a volume form and develop it in more generality.

The Volume Form D on a n-dimensional Vector Space

So let V be an n-dimensional vector space. Choose a basis $\mathscr{B} = \{b_1, \ldots, b_n\}$ for V. Then a volume form D on V is a function of n-tuples of vectors (n-hypervectors) $D(v_1, v_2, \ldots v_n)$ satisfying (see Lectute 14, page 2):

(i) $D(b_1, b_2, \ldots, b_n) = 1$.

(ii) $D(v_1, v_2, \ldots, v_n)$ is separately linear in each v_i when the other v_j's are left fixed.

(iii) If v_i and v_j are interchanged then the sign of D changes.

(iv) if v_i is replaced by $v_i + cv_j$, D does not change.

(v) If v_i is replaced by cv_i then D is multipled by c.

In the text, D is constructed and used to construct the determinant. We work in reverse. Let $v = (v_1, v_2, \ldots, v_n)$. Then we define

$$D(v_1, v_2, \ldots, v_n) = \det\left(P_{\mathscr{B} \longleftarrow \mathscr{V}}\right)$$

Theorem

This definition satisfies (i)-(v).

To explain what this means

$$A = (P_{\mathscr{B} \longleftarrow \mathscr{V}})$$

is given

$$A = (a_{ij}),$$

where

$$v_j = \sum_{i=1}^{n} a_{ij} b_i, \quad 1 \leq j \leq n \quad (*)$$

In terms of Lecture 13, we have the equality of hypervectors

$$(v_1, v_2, \ldots, v_n) = (b_1, b_2, \ldots, b_n) \cdot A \quad (**)$$

Then $D(v_1, v_2, \ldots, v_n) = \det(A)$.

In the above, $\mathscr{V}$ does not have to be a basis. For example we could take $V = \mathbb{R}^2$, $\mathscr{B} = \mathscr{E} = \{e_1, e_2\}$ and $\mathscr{V} = (e_1 + e_2, e_1 + e_2)$. Then

$$P_{\mathscr{B} \longleftarrow \mathscr{V}} = \begin{pmatrix} 1 & 1 \\ \ldots \end{pmatrix} \text{ and } \det(\mathscr{V}) = 0$$

$V = \mathbb{R}^2$, $\mathscr{B} = \{e_1, e_2\}$.

Then

$$(v_1, v_2) = \det \begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - dc$$

Here is the rule for $D(v_1, v_2, \ldots, v_n)$ for $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ and $\mathscr{B} = \{e_1, e_2, \ldots e_n\}$.
Put

$$A = \begin{pmatrix} v_1 & v_2 & \ldots & v_n \\ \downarrow & \downarrow & \ldots & \downarrow \end{pmatrix}$$

so A is the matrix whose [?] We always take the standard basis for $\mathscr{B}$ in case $V = \mathbb{R}^n$.

In fact $|D(v_1,\, v_2,\, \ldots,\, v_n)|$ is the volume of the solid in $\mathbb{R}^n$ which is the set of all combinations

$$\{t_1 v_1 + \ldots + t_n v_n : 0 \leq t_i \leq 1,\, 1 \leq i \leq n\}\,.$$

This is the generalized parallelepiped spanned by v_1, v_2, $\ldots$, v_n.