Lecture 22

Three examples/ problems

Problem 1 Show that the matrix $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ cannot be diagonalized.

Solution The characteristic polynomial $h(x)$ of A is given by $h(x) = (x-1)^2$. So the only possible eigenvalue of A is 1. The first standard basis vector $e_1 = (1,0)$ is an eigenvector belonging to 1, $Ae_1 = 1e_1$. Suppose there is another, say v. Put $P = \begin{array}{c} \begin{matrix} e_1 & v \end{matrix} \\ \begin{pmatrix} \downarrow & \downarrow \end{pmatrix} \end{array}$.

Then
$$P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

So
$$A = P\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}P^{-1} = PIP^{-1} = PP^{-1} = I$$

So
$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This is a contradiction so v does not exist.

Example, Problem 2

Diagonalize the symmetric matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solution

Step 1 Find the eigenvalues of A

Equivalently, find the roots of the characteristic polynomial

$$h(x) = \det(xI - A).$$

In fact it is often easier to work with $\det(A - xI) = (-1)^a h(x)$ where A is a by a.. So same roots. Now

$$\det(A - xI) = \det\begin{pmatrix} 2-x & 1 \\ 1 & -x \end{pmatrix}$$

$$= x^2 - 2x - 1$$

We apply the quadratic formula to solve the quadratic equation

$$x^2 - 2x + 1 = 0$$

Then by the quadratic formula the eigenvalues of A = the roots of x^2-2x-1 are $1 \pm \sqrt{2}$

Step 2. Find an eigenvector v_1 that belongs to the eigenvalue $1+\sqrt{2}$

So we must solve $A\begin{pmatrix} x \\ y \end{pmatrix} = (1+\sqrt{2})\begin{pmatrix} x \\ y \end{pmatrix}$ (*)

for x and y

Warning There are infinitely many solutions to (*) (all colinear).
But we just need ONE nonzero solution.

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (1+\sqrt{2})\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1+\sqrt{2})x \\ (1+\sqrt{2})y \end{pmatrix} \quad (*)$$

So

$$\begin{pmatrix} 2x+y \\ x \end{pmatrix} = \begin{pmatrix} (1+\sqrt{2})x \\ (1+\sqrt{2})y \end{pmatrix} \quad (*)$$

So we have two equivalent (same solutions) equations

$$2x+y = (1+\sqrt{2})x$$
$$x = (1+\sqrt{2})y$$

The second equation is easier to solve. Just put $y=1$ and get $x = 1+\sqrt{2}$

So $v_1 = (1+\sqrt{2}, 1)$

is a solution of the second equation and hence of both equations.

So we have found an eigenvector v_1 that belongs to the eigenvalue $1+\sqrt{2}$

<u>Step 3</u> Find an eigenvector v_2 that belongs to the eigenvalue $1-\sqrt{2}$. So we must solve

$$A\begin{pmatrix} x \\ y \end{pmatrix} = (1-\sqrt{2})\begin{pmatrix} x \\ y \end{pmatrix} \quad (**)$$

So

$$\begin{pmatrix} 2x+y \\ x \end{pmatrix} = \begin{pmatrix} (1-\sqrt{2})x \\ (1-\sqrt{2})y \end{pmatrix} \quad (**)$$

So

$$\begin{aligned} 2x+y &= (1-\sqrt{2})x \\ x &= (1-\sqrt{2})y \end{aligned} \quad (**)$$

Again we solve the second equation

$$x = (1-\sqrt{2})y.$$

Again put $y=1$ to find $x = 1-\sqrt{2}$.

So $V_2 = (1-\sqrt{2}, 1)$ is an eigenvector of A belonging to the eigenvalue $1-\sqrt{2}$.

Step 4 Write down the matrix P with columns v_1 and v_2

Write V_1 and V_2 as column vectors

$$V_1 = \begin{pmatrix} 1+\sqrt{2} \\ 1 \end{pmatrix} \quad \text{and} \quad V_2 = \begin{pmatrix} 1-\sqrt{2} \\ 1 \end{pmatrix}$$

Define the matrix P of eigenvectors by

$$P = \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

Step 5 Invert the matrix P of eigenvectors

Recall there is a simple formula for the inverse of any 2 by 2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Namely $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

(so flip the ~~diagonals~~ diagonals and change the signs of the off-diagonals, then divide by the determinant)

Apply this formula to P

using
$$\det(P) = 2\sqrt{2}$$

to obtain

$$P^{-1} = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -1+\sqrt{2} \\ -1 & 1+\sqrt{2} \end{pmatrix}.$$

The Final Step 6

Now we can diagonalize $A = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

Theorem

$$P^{-1}AP = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix}$$

Proof

$$P^{-1}AP = \frac{1}{2\sqrt{2}} \begin{pmatrix} 1 & -1+\sqrt{2} \\ -1 & 1+\sqrt{2} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 1+\sqrt{2} & 1 \\ -1+\sqrt{2} & -1 \end{pmatrix} \begin{pmatrix} 1+\sqrt{2} & 1-\sqrt{2} \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} 4+2\sqrt{2} & 0 \\ 0 & -4+2\sqrt{2} \end{pmatrix} = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & 1-\sqrt{2} \end{pmatrix} \quad \square$$

Finding an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A

Lemma v_1 and v_2 are orthogonal, that is $(v_1, v_2) = 0$.

Proof $(v_1, v_2) = ((1+\sqrt{2}, 1), (1-\sqrt{2}, 1))$

$$= (1+\sqrt{2})(1-\sqrt{2}) + (1)(1) = -1 + 1 = 0 \quad \square$$

Remark For any real symmetric matrix if v_1 and v_2 are eigenvectors belonging to ~~two~~ ~~different~~ eigenvalues ~~then~~ ~~they~~ are orthogonal.

We find $\|v_1\| = \sqrt{(1+\sqrt{2})^2 + 1^2} = \sqrt{4+2\sqrt{2}}$

$$\|v_2\| = \sqrt{(1-\sqrt{2})^2 + 1^2} = \sqrt{4-2\sqrt{2}}$$

Hence if we put $u_1 = \dfrac{1}{\sqrt{4+2\sqrt{2}}} v_1$ and $u_2 = \dfrac{1}{\sqrt{4-2\sqrt{2}}} v_2$ then $\mathcal{B} = \{u_1, u_2\}$ is an orthonormal basis for $\mathbb{R}^2$ consisting of eigenvectors of A.

Example/Problem 3

Diagonalize thee matrix $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Step 1 Find the eigenvalues of A

$$h(x) = \det(xI - A)$$

$$= \det \begin{pmatrix} x & 0 & -1 \\ 0 & x-1 & 0 \\ -1 & 0 & x \end{pmatrix}$$

expanding by the top row

$$= x(x-1)x - 1(-1)(-1)(x-1)$$

$$= x^2(x-1) - (x-1)$$

$$= (x-1)(x^2-1)$$

$$= (x-1)^2(x+1)$$

So the roots of the characteristic polynomial are

$+1, +1, -1$

Remark Since $A^2 = I$ the eigenvalues of A have to be either $+1$ or -1.

Why?

<u>Step 2</u> Find the eigenvector(s) of A belonging to the eigenvalue $+1$ (since $+1$ has multiplicity 2 as a root of the characteristic polynomial it is possible it has multiplicity 2 as an eigenvalue).

So we must solve $A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

So

$$\begin{pmatrix} z \\ y \\ x \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{or} \quad x = z \qquad (*)$$

The general solution of $(*)$ (the $+1$ eigenspace of A) $E_1(A)$ is given by

$$E_1(A) = \{(x, y, x) : x, y \in \mathbb{R}\}$$

So it is two dimensional and the eigenvalue $+1$ of A has multiplicity 2.

A basis for $E_1(A)$ is given by $\{(1,0,1), (1,1,1)\}$ (so $y=0$, then $y=1$)

So $(1,0,1)$ and $(1,1,1)$ are two independent eigenvectors belonging to the eigenvalue $+1$.

Step 3 Find an eigenvector of A belonging to the eigenvalue -1. So we must solve

$$A\begin{pmatrix}x\\y\\z\end{pmatrix} = -\begin{pmatrix}x\\y\\z\end{pmatrix}$$

So

$$\begin{pmatrix}z\\y\\x\end{pmatrix} = \begin{pmatrix}-x\\-y\\-z\end{pmatrix} \quad \text{so } y=0,\ z=-x$$

So $\begin{pmatrix}1\\0\\-1\end{pmatrix}$ is an eigenvector of A belonging to the eigenvalue -1.

Step 4 Write down the matrix P whose columns are the eigenvectors of A.

So

$$P = \begin{pmatrix}1 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & -1\end{pmatrix}$$

Step 5 Invert the matrix P

(next page)

$$\begin{array}{cc} P & I \\ \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{array}$$

$$\Big\downarrow \begin{array}{l} C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{array}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\Big\downarrow C_1 \to C_1 + \tfrac{1}{2} C_3$$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix} \qquad \begin{pmatrix} \frac{1}{2} & -1 & -1 \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & 1 \end{pmatrix}$$

$$\Big\downarrow C_3 \to -\tfrac{1}{2} C_3$$

$$\begin{array}{cc} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} & \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \\ I & P^{-1} \end{array}$$

__Step 6__ Check that the formula for P^{-1} is correct

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \checkmark$$

__Step 7__ Check that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

$$\begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad \checkmark$$