Lecture 5: More on Linear Transformations

Today, we tidy up some odds and ends.

Theorem (Text, Theorem 13.1)

Let $\mathscr{B} = \{b_1, \ldots, b_n\}$ be a basis for V. Let $w_1, \ldots, w_n$ be arbitrary vectors in W. Then there exists a unique $T \in \mathrm{Hom}(V, W)$ with

$$T(b_i) = w_i, \quad 1 \leq i \leq n.$$

Proof. Uniqueness is clear.
Existence: Define $T(v)$ for $v \in V$ as follows. Write $v = x_1 b_1 + \ldots x_n b_n$ and "define"

$$T(v) = \sum_{i=1}^{n} x_i w_i.$$

Is T well-defined?
Yes. Since $\mathscr{B}$ is a basis, the x_i's are uniquely determined. Also, note that T is linear. □

Now we can prove:

Theorem

Fix a basis $\mathscr{B} = (b_1, \ldots, b_n)$ for V. Then the map $M_{\mathscr{B}} : \mathrm{Hom}(V, V) \longrightarrow M_n(\mathbb{R})$ sending a linear transformation T to its matrix with respect to the basis $\mathscr{B}$ is 1:1 and onto.

Proof.

1:1: Let $T_1, T_2 \in \mathrm{Hom}(V, V)$. Suppose $M_{\mathscr{B}}(T_1) = M_{\mathscr{B}}(T_2) = (a_{ij})$.

Then for $1 \le j \le n$, $T_1(b_j) = \sum_{i=1}^{n} a_{ij} b_i$ and

$T_2(b_j) = \sum_{i=1}^{n} a_{ij} b_i$.

Thus, $T_1(b_j) = T_2(b_j)$ hence $T_1 = T_2$.

onto: Let $(a_{ij}) \in M_n(\mathbb{R})$.

Then define $w_j = \sum_{i=1}^{n} a_{ij} b_i$, $1 \le j \le n$.

There exits T with $T(b_i) = w_i$, $1 \le i \le n$. □

Back to $T : V \longrightarrow W$

There are two useful subsets.

Definition

The **Range** of T, denoted $T(V)$ or $R(T)$ is defined as

$$T(V) := \{T(v) : v \in V\} \subset W.$$

Lemma: $T(V)$ is a subspace.
Proof. Use the fact that T is linear.

Definition

The **Nullspace** of T, denoted $N(T)$ is defined as

$$N(T) := \{v \in V : T(v) = 0\} \subset V.$$

Lemma: N(T) is a subspace.
Proof. Use the fact that T is linear.

A dimension formula

Theorem (Text, Theorem 13.9)

Let $T \in \mathrm{Hom}(V, W)$. *Then*

$$\dim T(V) + \dim N(T) = \dim V$$

To prove this, we will need a very useful proposition.

Proposition

Any linearly independent set $\{v_1, \ldots, v_k\} \in V$ *can be completed to a basis* $\{v_1, \ldots, v_k, v_k + 1, \ldots, v_n\}$ *for* V*.*

Proof. Assume $\dim V = n$. If $\{v_1, \ldots, v_k\}$ spans V we are done. Otherwise there exists a vector v_{k+1} that is not in the span of $\{v_1, \ldots, v_k\} \in V$. Then $\{v_1, \ldots, v_k, v_k + 1\}$ is still independent. If $\{v_1, \ldots, v_k, v_k + 1\}$ spans V, we are done, otherwise, continue. We will be done in $n - k$ steps.

Remark: Any spanning set can be refined to a basis.

Proof of Theorem 13.9. Assume $\dim V = n$ and $\dim N(T) = k$. Choose a basis $\{b_1, \ldots, b_k\}$ for $N(T)$ and complete it to a basis $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ for V. It sufficed to prove
Claim: $\{T(b_{k+1}, \ldots, T(b_n)\}$ is a basis for $R(T)$.
Spanning set: Clear.
Independent set: Suppose $x_{k+1}T(b_{k+1} + \ldots + x_nT(b_n) = 0$. Then $x_{k+1}b_{k+1} + \ldots + x_nb_n \in N(T)$ and hence $x_{k+1}b_{k+1} + \ldots + x_nb_n = x_1b_1 + \ldots + x_kb_k$.
Thus

$$x_1b_1 + \ldots + x_kb_k - x_{k+1}b_{k+1} - \ldots - x_nb_n = 0$$

But $\{b_1, \ldots, b_k, b_{k+1}, \ldots, b_n\}$ is a basis, so all the coefficients x_i, $1 \leq i \leq n$ are zero. Hence $x_{k+1}, \ldots, x_n$ are zero. □

First, we need

Proposition

Suppose $T : V \longrightarrow W$. *Then* T *is 1:1* $\Longleftrightarrow N(T) = \{0\}$.

Proof. ($\Longrightarrow$) Suppose $T(v) = 0$. Then since $T(0) = 0$, we have $v = 0$, hence if $v \in N(T)$ then $v = 0$.
($\Longleftarrow$) Suppose $T(v_1) = T(v_2)$. Then since T is linear $T(v_1 - v_2) = 0$, hence $v_1 - v_2 \in N(T)$ and thus $v_1 - v_2 = 0$. Finally, $v_1 = v_2$. $\square$

Consequences

Proposition

Let $T \in \mathrm{End}(V)$ *(so* $V = W$*). Then* T *is 1:1* $\Longleftrightarrow$ T *is onto.*

Proof. We use $\dim V = \dim R(T) + \dim N(T)$.
($\Longrightarrow$) T is 1:1 so $N(T) = 0$, hence $\dim R(T) \dim V$, thus $R(T) = V$ (since $R(T) \subset V$).
($\Longleftarrow$) T is onto, so $R(T) = V$ and $\dim R(T) = \dim V$. Thus $\dim N(T) = 0$ and so $N(T) = \{0\}$.

Warning: This is not true if $W \neq V$.

Now do problem pg. 108 # 10.