Lecture 6: The matrix of a Linear Transformation Relative to a Basis

Let V be a vector space and $\mathscr{B} = \{b_1, \ldots, b_n\}$ be a basis for V. Then any v has unique coordinates $[v]_{\mathscr{B}} = (x_1, \ldots, x_n)$ relative to $\mathscr{B}$ defined by

$$v = \sum_{i=1}^{n} x_i b_i$$

Now let V and W be vector spaces and $T : V \longrightarrow W$ be a linear transformation. Suppose $\mathscr{B} = \{b_1, \ldots, b_n\}$ is a basis for V and $\mathscr{C} = \{c_1, \ldots, c_n\}$ is a basis for W.

Definition

The matrix of the linear transformation T relative to the basis $\mathscr{B}$ and $\mathscr{C}$ and written ${}_{\mathscr{C}}[T]_{\mathscr{B}}$ is the $m \times n$ matrix (a_{ij}) given by

$$T(b_j) = \sum_{i=1}^{m} a_{ij} c_i, \quad 1 \leq j \leq n, \qquad (*)$$

It is important to understand the physical meaning of $(*)$. The first column of A is the coordinates of $T(b_i)$ relative to $c_1, \ldots, c_m$; the second column of A is the coordinates of $T(b_2)$ relative to $c_1, \ldots, c_m$, etc.

So

$$A\begin{pmatrix} T(b_1) & T(b_2) & \dots & T(b_n) \\ \downarrow & \downarrow & & \downarrow \\ & & & \end{pmatrix}$$

or

$$A\begin{pmatrix} [T(b_1)]_{\mathscr{C}} & [T(b_2)]_{\mathscr{C}} & \dots & [T(b_n)]_{\mathscr{C}} \\ & & & \\ & & & \end{pmatrix}$$

We will usually have $V = W$ and $\mathscr{B} = \mathscr{C}$. Then ${}_{\mathscr{B}}[T]_{\mathscr{B}}$ is a square matrix.

If thre is only one basis present we will write $M(T)$ instead of ${}_{\mathscr{B}}[T]_{\mathscr{B}}$.

Problem

Let $\mathrm{Pol}_3(\mathbb{R})$ be the polynomial functions of degree less than or equal to 3. Let $\frac{\mathrm{d}}{\mathrm{d}x} : \mathrm{Pol}_3(\mathbb{R}) \longrightarrow \mathrm{Pol}_3(\mathbb{R})$ be differentiation. Compute the matrix ${}_{\mathscr{B}}\left[\frac{\mathrm{d}}{\mathrm{d}x}\right]_{\mathscr{B}} = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ relative to $\mathscr{B} = \{x_1, x, x^2, x^3\}$.

Solution

$\frac{\mathrm{d}}{\mathrm{d}x}(1) =$ the zero polynomial $= (0, 0, 0, 0)$ so the first column of $M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ is

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

$\frac{\mathrm{d}}{\mathrm{d}x}(x) = 1 = (1,\ 0,\ 0,\ 0)$ so the second column of $M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ is $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x = (0)1 + 2(x) + 0(x^2) + 0(x^3) = (0,\ 2,\ 0,\ 0)$ so the third column of $M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)$ is $\begin{pmatrix} 0 \\ 2 \\ 0 \\ 0 \end{pmatrix}$.

Finally $\frac{\mathrm{d}}{\mathrm{d}x}(3^2) = 3x^2 = (0,\ 0,\ 3,\ 0)$.

We obtain:

$$
{}_{\mathscr{B}}\left[\frac{\mathrm{d}}{\mathrm{d}x}\right]_{\mathscr{B}} = M\left(\frac{\mathrm{d}}{\mathrm{d}x}\right) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

Notation: Let $\bullet$ denote matrix multiplication. Then we have the important

Proposition

[1] Let U, V, W be vector spaces with basis $\mathscr{A} = \{a_1, \ldots, a_m\}$, $\mathscr{B} = \{b_1, \ldots, b_n\}$ and $\mathscr{C} = \{c_1, \ldots, c_p\}$ respectively. Let $T : U \longrightarrow V$ and $S : V \longrightarrow W$ be linear transformations. Then

$$_{\mathscr{C}}[S \circ T]_{\mathscr{A}} = {}_{\mathscr{C}}[S]_{\mathscr{B}} \bullet {}_{\mathscr{B}}[T]_{\mathscr{A}}$$

Proof. Put

$$\begin{aligned} Z &= (z_{ik}) = M\,(S \circ T) \\ Y &= (y_{jk}) = M\,(T) \\ X &= (x_{ij}) = M\,(T) \end{aligned}$$

[put diagram here]
We will compute $(S \circ T)(a_k)$ in two ways.
The matrix (z_{ik}) is defined by

$$(S \circ T)(a_k) = \sum_{i=1}^{p} z_{ik} c_i.$$

Now we compute $(S \circ T)(a_k)$ another way. We have

$$(S \circ T)(a_k) = S(T(a_k)) \qquad (*)$$

But the matrix $Y = (y_{jk})$ is defined by

$$T(a_k) = \sum_{i=1}^{n} y_{ik} b_j . \qquad (**)$$

We substitute the RHS of $(**)$ into $(*)$ to get

$$\begin{aligned}(S \circ T)(a_k) &= S(T(a_k)) = S\left(\sum_{i=1}^{n} y_{ik} b_j\right) \\ &= \sum_{i=1}^{n} y_{ik} S\left(b_j\right) \qquad (\#).\end{aligned}$$

But the matrix $X = (x_{ij})$ is defined by

$$S(b_j) = \sum_{i=1}^{p} x_{ij} c_i.$$

We substitute $(\#\#)$ into $(\#)$ to get

$$\begin{aligned}
(S \circ T)(a_k) &= \sum_{j=1}^{n} y_{ik} S\left(\sum_{i=1}^{p} x_{ij} c_i\right) \\
&= \sum_{j=1}^{n} \sum_{i=1}^{p} y_{jk} x_{ij} c_i \\
&= \sum_{i=1}^{p} \left(\sum_{j=1}^{n} x_{ij} y_{jk} c_i\right).
\end{aligned}$$

Hence

$$\sum z_{ik} c_i = \sum_{i=1}^{p} \left(\sum_{j=1}^{n} x_{ij} y_{jk} c_i \right).$$

Since c_i is a basis for W, we have

$$z_{ik} = \sum_{j=i}^{n} x_{ij} y_{jk}.$$

But the RHS is the ik-th entry of the product matrix $X \bullet Y$.
Remark: This wouldn't have worked if we had written the vectors $T(b_j)$ along the rows instead along the columns.

Proposition

Let V be a vector space and $T \in L(V\,V) = \mathrm{Hom}\,(V,\,V)$. Let $\mathscr{B} = \{b_1,\,\ldots,\,b_n\}$ be a basis for V. Let $v \in V$. Then

$$[T(v)\mathscr{B}]_{=\ \mathscr{B}}[T]_{\mathscr{B}}\,[v]_{\mathscr{B}}\,.$$

Proof. Suppose ${}_{\mathscr{B}}[T]_{\mathscr{B}} = A = (a_{ij})$ and $[v]_{\mathscr{B}} = (x_1,\,x_2,\,\ldots,\,x_n)$ so

$$v = \sum_{j=1}^{n} x_j b_j.$$

Then

$$T(v) = \sum_{j=1}^{n} x_j T(b_j). \qquad (*)$$

But by definition of the matrix ${}_{\mathscr{B}}[T]_{\mathscr{B}}$

$$T(b_j) = \sum_{i=1}^{n} a_{ij} b_i (**)$$

Subsitute $(**)$ into $(*)$ to obtain

$$\begin{aligned} T(v) &= \sum_{j=1}^{n} x_j \left(\sum_{i=1}^{n} a_{ij} b_i \right) \\ &= \sum_{i=1}^{n} \sum_{i=1}^{n} x_j a_{ij} b_i \\ &= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x_j \right) b_i \end{aligned}$$

Hence

$$[T(v)]_{\mathscr{B}} = \left(\sum_{j=1}^{n} a_{1j} x_j, \ldots, \sum_{j=1}^{n} a_{nj} x_j \right).$$

But

$$\begin{aligned} A\,[v]_{\mathscr{B}} &= \begin{pmatrix} a_{11} & \ldots & a_{1n} \\ a_{21} & \ldots & a_{2n} \\ \vdots & & \vdots \\ a_{n1} & \ldots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \begin{pmatrix} \sum_{j=1}^{n} a_{1j} x_j \\ \vdots \\ \sum_{j=1}^{n} a_{nj} x_j \end{pmatrix} \end{aligned}$$

At this stage we are not differentiating between row vectors and column vectors.

Isomorphism of Algebras

Theorem (Text, Theorem 13.3)

Suppose $\dim V = n$ *and* $\{b_1, \ldots, b_n\}$ *is a basis for* V. *Then the map*

$$M : \mathrm{Hom}\,(V, V) \longrightarrow M_n(\mathbb{R})$$

that sends T *to* $M(T)$ *is 1:1, onto, linear and send compositions* $\circ$ *of linear transformations to multiplications* $\bullet$ *of matrices* M *is said to be an isomorphism of algebras.*

Future Problem

<u>Problem</u>

Suppose $T : V \longrightarrow V$ and $\mathscr{B}$ and $\mathscr{C}$ are bases for V.
How are matrices ${}_{\mathscr{B}}[T]_{\mathscr{B}}$ and ${}_{\mathscr{C}}[T]_{\mathscr{C}}$ related?
This problem will be addressed in Lecture 8.