Lecture 10: The Gram-Schmidt Orthogonalization Process

Let $\{v_1, \ldots, v_n\}$ be an ordered independent set. Then there exists an orthonormal set $\{u_1, \ldots, u_k\}$ such that

$$\text{span}\,\{v_1, \ldots, v_n\} = \text{span}\,\{u_1, \ldots, u_k\}, \quad 1 \leq i \leq k.$$

Proof.

Step 1

Put $u_1 = \dfrac{v_1}{||v_1||}$.

Step 2

Make v_2 orthogonal to u_1 by defining

$$w_2 = v_2 - (v_2,\, u_1)\, u_1.$$

Remark: In lecture 11, we will learn that $(v_2,\, u_1)\, u_1$ is the projection of v_2 onto the line through v_1 so we are subtracting this projection from v_1. Then w_2 is perpendicular to u_1. Indeed

$$
\begin{aligned}
(w_2,\, u_1) &= (v_2 - (v_2,\, u_1)\, u_1,\, u_1) \\
&= (v_2,\, u_1) - (v_2,\, u_1)\,(u_1,\, v_1) \\
&= (v_2,\, u_1) - (v_2,\, u_1) \\
&= 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathrm{span}\,(u_1,\, w_2) &= \mathrm{span}\,(v_1,\, w_2) \\
&= \mathrm{span}\,(v_1,\, v_2)\,. \quad (**)
\end{aligned}
$$

$(**)$ holds because any linear combination of v_1 and w_2 is a linear combination of v_1 and v_2 and vice versa.

Here is why:

$v \in \operatorname{span}(v_1, w_2) \Longrightarrow$ there exists c_1 and c_2 so that

$$v = c_1 v_1 + c_2 v_2.$$

But

$$\begin{aligned} w_2 &= v_2 - (v_2, u_1) u_1 \\ &= v_2 - \frac{1}{||v_1||^2} (v_2, v_1) v_1. \end{aligned}$$

So

$$\begin{aligned} v &= c_1 v_1 + c_2 \left(v_2 - \frac{1}{||v_1||^2} (v_2, v_1) v_1 \right) \\ &= \left[c_1 - \frac{1}{||v_1||^2} (v_2, v_1) \right] v_1 + c_2 v_2. \end{aligned}$$

So, $v \in \operatorname{span}(v_1, v_2)$. Conversely, if $v \in \operatorname{span}(v_1, v_2)$. Write

$$v_2 = w_2 + \frac{1}{||v_1||^2} (v_2, v_1) v_1$$

and procede as above to prove $v \in \operatorname{span}(v1, w_2)$.

At each stage in the following prood a formula like $(**)$ has to be checked. I will leave this to you.
Now we have $\{u_1, u_2, v_3, v_4, \ldots, v_k\}$ with $\{u_1, u_2\}$ and orthonormal set.
Now make v_3 orthogonal by defining

$$w_3 = v_3 - (v_3, u_1)\, u_1 - (v_3, u_2)\, u_2.$$

(We are subtracting off the projection of v_3 onto $\operatorname{span}\{u_1, u_2\}$)
We have as before (only more complicated)

The induction step from $i-1$ to i

Suppose we have

$$\{u_1\, u_2,\, \ldots,\, u_{i-1},\, v_i,\, v_{i+1},\, \ldots,\, v_k\}$$

where $\{u_1\, u_2,\, \ldots,\, u_{i-1}\}$ is an orthonormal set with

$$\operatorname{span}\{u_1\, u_2,\, \ldots,\, u_{i-1}\} = \operatorname{span}\{v_1\, v_2,\, \ldots,\, v_{i-1}\}.$$

Put $w_i = v_i - [(v_i,\, u_1)\, u_1 + \ldots (v_i,\, u_{i-1})\, u_{i-1}]$.
Then $(w_i,\, u_j) = 0, \quad 1 \le j \le i-1$ and

$$\begin{aligned} \operatorname{span}\{u_1\, u_2,\, \ldots,\, u_{i-1}, w_i\} &= \{u_1\, u_2,\, \ldots,\, u_{i-1}, v_i\} \\ &= \operatorname{span}\{v_1\, v_2,\, \ldots,\, v_{i-1}, v_i\}. \end{aligned}$$

Now, put

$$u_i = \frac{w_i}{||w_i||}$$

and we have performed the induction step. □

Corollary

Every finite dimensional vector space V with an inner product has an orthonormal basis.

Proof. Choose a basis $\mathscr{B} = \{b_1, \ldots, b_n\}$ for V and apply Gram-Schmidt to $\mathscr{B}$ to get an orthonormal basis for V.

Problem

Hard Problem: Show that change of basis matrix $P_{\mathscr{B}\longleftarrow\mathscr{U}}$ from $\mathscr{U}$ to $\mathscr{B}$ is upper triangular, that is

$$P_{\mathscr{B}\longleftarrow\mathscr{U}} = \begin{pmatrix} * & * & \dots & * \\ 0 & * & \dots & * \\ 0 & 0 & \dots & * \\ \vdots & \vdots & \vdots & * \\ 0 & 0 & \dots & 0 \end{pmatrix}$$

How is this related to

$$\operatorname{span}\{u_1, u_2, \dots, v_i\} = \operatorname{span}\{b_1, b_2, \dots, b_i\}, \ 1 \le i \le n?$$