Lecture 13: The Existence of the Determinant

I am not going to follow the text but I will prove all the results in the text (Chapter 5).

You already know the formula for the determinant of a 2 by 2 matrix

$$\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

There is also a simple formula for the determinant of 3 by 3 matrix.

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ h & i & j \end{pmatrix} = hec + dbj + aif - aej - bfh - cdi.$$

The first row in the formula goes diagonally $\backslash$ and the second row goes $/$.

In this lecture we will prove the following

Theorem

There exits a unique real-valued $M_n(\mathbb{R})$ of n by n matrices satisfying

(i) $\det(I_n) = 1$.

(ii) $\det(A)$ *is linear in the rows of* A.

(iii') *If two rows of A are interchanged the determinant is multiplied by -1. This is a type 1 elementary row operation.*

(iv) *If the i-th row R_i is replaced by $R_i + cR_j$ where R_j is a different row $i \neq j$, then the determinant is unchanged. This is a type 2 elementary row operation.*

(v) *If the i-th row R_i is replaced by cR_i then the determinant is multiplied by c. This is a type 3 elementary row operation.*

Remark: There is a weaker version of (iii) which we will prove first:
(iii) If two rows of A are the same then $\det(A) = 0$.

Exercise: Prove that (iii) $\longrightarrow$ (iii'). Hint: Interchange the two rows that are equal.

Example of Axiom (ii)

We give an example of what (ii) says

(ii) Linear in the Second Row

$$\det \begin{pmatrix} 1 & 2 & 4 \\ 2 \cdot 5 & 2 \cdot 6 & 2 \cdot 3 \\ 2 & -1 & 0 \end{pmatrix} = 2 \cdot \det \begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 3 \\ 2 & -1 & 0 \end{pmatrix}$$

and

$$\det \begin{pmatrix} 1 & 2 & 4 \\ 3+5 & 4+6 & 2+3 \\ 2 & -1 & 0 \end{pmatrix} = \det \begin{pmatrix} 1 & 2 & 4 \\ 3 & 4 & 2 \\ 2 & -1 & 0 \end{pmatrix} + \det \begin{pmatrix} 1 & 2 & 4 \\ 5 & 6 & 3 \\ 2 & -1 & 0 \end{pmatrix}$$

So if you fix $n-1$ rows $\det$ is a linear function of the remaining row.

Proof of Theorem

We now prove the theorem.

First we prove uniqueness.

Case 1

A has rank n. Then we can find a sequence of elementary row operations $E_1, E_2 \ldots, E_k$ such that

$$(E_1, E_2 \ldots, E_k)(A) = I_n$$

Since we know how he determinant changes each time we apply an elementary row operation and we know $\det(I_n) = 1$, we find that $\det(A)$ is unique in case A has full rank.

Case 2

Suppose rank $A < n$. Then we can find a sequence of elementary row operations $E_1, E_2 \ldots, E_m$ so that

$$(E_1, E_2 \ldots, E_m)(A) = B,$$

where the last row of B is the zero row.

But if a matrix B has a row R of zeroes, then we can use axiom (v) to show

$$\det(B) = 0.$$

Let E be the operation that multiplies row R by -1. Then by axiom (v)

$$\det(E(B)) = -\det(B).$$

But since - 0 = 0, $E(B)$ is the same matrix as B so

$$\det(E(B)) = \det(B).$$

Hence $\det(B) = -\det B$ so $\det(B) = 0$.

Example 1

The determinant of a diagonal matrix

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

We apply axiom (v) three times to obtain

$$\det(A) = \lambda_1 \lambda_2 \lambda_3 \det(I_3).$$

Hence by axiom (i) we have

$$\det(A) = \lambda_1 \lambda_2 \lambda_3$$

and the determinant of a diagonal matrix is the product of its diagonal entries.

Example 2

The determinant of an upper triangular matrix is the product of its diagonal entries

$$A = \begin{pmatrix} \lambda_1 & a & h \\ 0 & \lambda_2 & c \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

If $\lambda_3 = 0$ then $\det(A) = 0$. If λ_3 is not zero, do two elementary operations of type 2 to get

$$A = \begin{pmatrix} \lambda_1 & a & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

If $\lambda_2 = 0$, then $\det(A) = 0$. Otherwise, do another elementary row operations of type 2 to get

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

Existence of the Determinant

Let A_{ij} be the submatrix obtained from A by deleting the i-th row and the j-th column.

We define $\det(A)$ inductively. Assume $\det(A)$ exists satisfying (i)-(v) for $n-1$ by $n-1$ matrices. Let A be an n by n matrix.

Definition

$$\det(A) = \sum_{i=1}^{n} (-1)^{i-1} a_{ij} \det(A_{i1}) \qquad (*)$$

so we are expanding by the first column.

We will now verify (i), (ii), (iii), (iv), (v). In fact (i) and (v) are clear by induction. We will prove (v) on pages 10-13.

We begin with (ii)–this is the hardest one.

Write

$$A = \begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix},$$

where R_1 is the first row, R_2 the second row, etc.

Suppose we multiply the first row by c so we get

$$B = \begin{pmatrix} cR_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix}.$$

Let B_{ij} be the result of removing the i-th row and the j-th column from B.

We must show

$$\sum_{i=1}^{n} (-1)^{i-1} b_{i1} \det(B_{i1}) = c \sum_{i=1}^{n} (-1)^{i-1} a_{i1} \det(A_{i1}) \qquad (**)$$

This is the same as axiom (v).

We use the following notation: if $R = (r_1, r_2, \ldots, r_n)$ is a row vector then $R^{(1)} = (r_2, r_3, \ldots, r_n)$.

We will show each term on the left of $(**)$ is c times its counterpart on the right; that is

$$b_{i1} \det (B_{i1} = ca_{i1} \det (A_{i1})$$

There are two cases, $i = 1$ and $i > 1$. We first do $i = 1$. We have $b_{11} = ca_{11}$.

Also

$$B_{11} = \begin{pmatrix} R_2^{(1)} \\ R_3^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix}.$$

Hence $\det (B_{11}) = \det (A_{11})$, so

$$b_{11} \det (B_{11}) = ca_{11} \det (A_{11})$$

Now look at the second term $b_{21} \det(B_{21})$ in $(**)$. Only the first row of A gets changed so

$$b_{21} = a_{21}.$$

Also

$$B_{21} = B_{11} = \begin{pmatrix} R_1^{(1)} \\ R_2^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix}, \text{ and } A_{21} = B_{11} = \begin{pmatrix} R_1^{(1)} \\ R_2^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix}.$$

Hence by induction

$$\det(B_{21}) = c \det(A_{21}).$$

We obtain

$$b_{21} \det(B_{21}) = c a_{21} \det(A_{21}).$$

The same argument works for all terms $b_{i1} \det(B_{i1})$ for all $i \geq 2$.

Also the same argument works if we put

$$B = \begin{pmatrix} cR_1 \\ \vdots \\ R_i \\ \vdots \\ R_n \end{pmatrix}.$$

To complete the proof of (ii) we have to prove that in

$$A = \begin{pmatrix} \vdots \\ R+S \\ \vdots \end{pmatrix} \text{ then}$$

$$\det(A) = \det \begin{pmatrix} \vdots \\ R \\ \vdots \end{pmatrix} + \det \begin{pmatrix} \vdots \\ S \\ \vdots \end{pmatrix}$$

Again, we do only the case of the first row. Put $R = (r_1, r_2, \ldots, r_n)$ and $S = (s_1, s_2, \ldots, s_n)$ so

$$A = \begin{pmatrix} R+S \\ R_2 \\ \vdots \\ R_n \end{pmatrix}.$$

Put

$$B = \begin{pmatrix} R \\ R_2 \\ \vdots \\ R_n \end{pmatrix} \text{ and } C = \begin{pmatrix} S \\ R_2 \\ \vdots \\ R_n \end{pmatrix}.$$

So we want to prove

$$\det(A) = \det(B) + \det(C).$$

We claim that we again get term-wise equality in the formula $(*)$. So we have to prove

$$a_{i1} \det(A_{i1}) = b_{i1} \det(B_{i1}) + c_{i1} \det c_{i1} \qquad (***)$$

We first take $i = 1$. Then we have

$$a_{11} = r_1 + s_1, \, b_{11} = r_1 \text{ and } c_{11} = s_1.$$

Also, $A_{11} = B_{11} = C_{11}$ because A, B, C coincide except for the first row. Hence the left-hand side in $(***)$ is $(r_1 + s_1)\det(A_{11})$ and the right-hand side is $r_1 \det(A_{11}) + s_1 \det(A_1 1)$ so the first terms of $(***)$ agree.

We now consider the second term of $(***)$ so $i = 2$. So we omit the **second** rows of A, B, C. Now we have

$$a_{21} = b_{21} = c_{21}$$

and

$$A_{21} = \begin{pmatrix} R^{(1)} + S^{(1)} \\ R_3^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix}$$

$$B_{21} = \begin{pmatrix} R^{(1)} \\ R_3^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix} \text{ and } S_{21} = \begin{pmatrix} R^{(1)} \\ R_3^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix}$$

Hence, the first row of A_{21} is the sum of the first row of B_{21} and the first row of C_{21}. Hence by induction

$$\det(A_{21}) = \det(B_{21}) + \det(C_{21})$$

The same argument works for all $i \geq 2$.

Linearity allows us to operate one row at a time with other rows fixed as we will see on page 19.

It remains to prove (iii) and (iv). We will show (iii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii).

We now prove (iii). We will do the special case where the first two rows are equal so

$$A = \begin{pmatrix} R \\ R \\ R_3 \\ \vdots \\ R_n \end{pmatrix}$$

Recall the definition of $\det$.

$$\det(A) = \sum_{i=1}^{n} (-1)^{i-1} a_{i1} \det(A_{i1}) \quad (*)$$

We will now show each term in $(*)$ is zero except for the first two terms. Indeed, if $i \geq 2$.

$$A = \begin{pmatrix} R^1 \\ R^1 \\ \vdots \\ R_n^1 \end{pmatrix}$$

then A_{i1} has the first two rows the same, so by induction

$$\det(A_{i1}) = 0, \quad i > 2.$$

Hence we are left with two terms in $(*)$

$$\det(A) = a_{11}\det(A_{11}) - a_{21}\det(A_{21})$$

and

$$A_{11} = A_{21} = \begin{pmatrix} R^{(1)} \\ R_3^{(1)} \\ \vdots \\ R_n^{(1)} \end{pmatrix}$$

Hence $a_{11}\det(A_{11}) = a_{21}\det(A_{21})$ and we have proved (iii).

We will now prove (iii) implies (iv). Again, we will do a special case. Suppose we add c times R_1 to R_2. Then, by (ii), we have

$$\det\begin{pmatrix} R_1 \\ R_2 + cR_1 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} + \det\begin{pmatrix} R_1 \\ cR_1 \\ \vdots \\ R_n \end{pmatrix}$$

But by axiom (v), we have

$$\det \begin{pmatrix} R_1 \\ R_1 \\ \vdots \\ R_n \end{pmatrix} = c \det \begin{pmatrix} R_1 \\ R_1 \\ \vdots \\ R_n \end{pmatrix}$$

and by (iii')

$$\det \begin{pmatrix} R_1 \\ R_1 \\ \vdots \\ R_n \end{pmatrix} = 0$$

Exercise: Do the general case.

Now we can prove (iii). We will prove that if we interchange R_1 and R_2 the determinant changes sign. We will apply (iv) four times and (ii) at the end.

$$\det\begin{pmatrix} R_1 \\ R_2 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_1 + (R2 - R1) \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix}$$

$$= \det\begin{pmatrix} R_1 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_2 \\ R_2 - R_1 \\ \vdots \\ R_n \end{pmatrix} = \det\begin{pmatrix} R_2 \\ -R_1 \\ \vdots \\ R_n \end{pmatrix}$$

Exercise: Do the general case.

We have proved that $\det$ exist.

Problem

What does the above proof have to do with the following identity?

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$