Lecture 15

Properties of the Determinant

Last time we proved the existence and uniqueness of the determinant. $\det : M_{n \times n}(F) \longrightarrow F$ satisfying 5 axioms.

For this lecture we will be using the last three axioms dealing with how $\det(A)$ behaves when elementary row operations are performed on A.

Type 1

Interchange two rows of A to get a new matrix B. Then we

$$\det(B) = -\det(A)$$

Type 2 Replace the i-th row R_i of A by $R_i + cR_j$ where R_j is a different row to get B. Then

$$\det(B) = \det(A)$$

Type 3

Multiply the i-th row R_i of A by c (and don't change the other rows) to get B. Then

$$\det(B) = c\det(A)$$

The key point in what follows is that each of the three operations can be realized by multiplying A on the left by an "elementary matrix".

Proposition 1

Let E be the result of applying one of the above elementary operations to the identity matrix and B be the result of applying the same operation to A. Then

$$B = EA$$

I won't prove this in general but I will prove it for $n=2$.

<u>Type 1</u> (<u>Interchange the two rows</u>)

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B=\begin{pmatrix} c & d \\ a & b \end{pmatrix}$$

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$EA=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} c & d \\ a & b \end{pmatrix}=B$$

so $$EA=B$$

<u>Type 2</u> (Add λR_1 to R_2)

$$A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B=\begin{pmatrix} a & b \\ c+\lambda a & d+\lambda b \end{pmatrix}$$

$$I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E=\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

$$EA=\begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}=\begin{pmatrix} a & b \\ c+\lambda a & d+\lambda b \end{pmatrix}=B.$$

Type 3 (Multiply the second row by λ)

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad B = \begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}$$

$$EA = \begin{pmatrix} 1 & 0 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ \lambda c & \lambda d \end{pmatrix} = B$$

In what follows I will let $\mathcal{E}(A)$ denote the result of applying an elementary row operation $\mathcal{E}$ to a matrix A and E be the elementary matrix $\mathcal{E}(I)$ so Proposition 1 can be restated as

$$\mathcal{E}(A) = \underbrace{\mathcal{E}(I)}_{E} \cdot A$$

We now compute $\det(E)$ for the three types of elementary matrices.

Lemma 1

Type 1

$$\det(E) = \det(\mathcal{E}(I)) = -\det(I) = -1$$

(the second equality holds by axiom (iii))

Type 2

$$\det(E) = \det(\mathcal{E}(I)) = \det(I) = 1$$

(the second equality holds by axiom (iv))

Type 3 ($\mathcal{E}$ = multiply R_i by c)

$$\det(E) = \det(\mathcal{E}(I)) = c\det(I) = c.$$

(the second equality holds by axiom (v))

We can now prove

Lemma 2

Let E be an elementary matrix and A be an arbitrary n by n matrix. Then

$$\det(EA) = \det(E)\det(A)$$

Proof: By Proposition 1 we have

$$EA = \mathcal{E}(A).$$

and so

$$\det(EA) = \det(\mathcal{E}(A)) \quad (*)$$

Now evaluate the RHS of $(*)$ by axioms (iii), (iv) and (v) to show that

$$\det(\mathcal{E}(A)) = \det(E)\det(A)$$

using Lemma 1 to evaluate $\det(E)$

For example if E interchanges two rows then in Lemma 1, we saw $\det(E) = -1$.

so

$$\det(E)\det(A) = (-1)\det(A) = -\det(A)$$

By axiom (iii)

$$\det(EA) = \det(\mathcal{E}(A)) = -\det(A)$$

$\square$

Lemma 3

Let $E_1, \dots, E_k$ be elementary matrices and B be an arbitrary n by n matrix. Then

$$\det(E_1 \cdots E_k B) = \det(E_1 \cdots E_k)\det(B)$$

Proof

By induction on k. We have already proved the case of $k=1$. Suppose it is true for k and consider

$$D = \det(E_1 \underbrace{E_2 \cdots E_k E_{k+1} B}_{A})$$

Put $A = E_2 E_k \cdots E_{k+1} B$ so the above determinant becomes $D = \det(E_1 A)$. But by Lemma 2 we have

$$\det(E_1 A) = \det(E_1)\det(A)$$

and by the induction hypothesis we have

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$$D = \det(E_1)\det(E_2 \cdots E_k E_{k+1} B)$$
$$= \det(E_1)\underbrace{\det(E_2 \cdots E_k E_{k+1})\det(B)}_{\text{by the induction hypothesis}}$$

$$= \det(E_1 E_2 \cdots E_k E_{k+1})\det(B)$$

The last equality is Lemma 2 with $A = E_2 \cdots E_k E_{k+1}$. $\square$

<u>Corollary</u>

$\det(E_1 \cdots E_k) = \det(E_1) \cdots \det(E_k) \neq 0$

<u>Proof</u>

$\det(E_1 E_2 \cdots E_k) = \det(E_1)\det(E_2 \cdots E_k)$
$= \det(E_1)\det(E_2) \cdots \det(E_k)$

The last equality follows by induction.

We now apply Lemma 3 to prove two important properties of determinants

Theorem 1

An n by n matrix A is invertible if and only $\det(A) = 0$.

Proof

Recall that A can be reduced by elementary row operations to a matrix B which is the identity if A is invertible and has a row of zeros if A is not invertible. So we have

$$E_1 E_2 \cdots E_k A = B$$

and hence

$$\det(E_1 E_2 \cdots E_k A) = \det(B).$$

By Lemma 3 we have

$$\det(E_1 E_2 \cdots E_k) \det(A) = \det(B)$$

By the previous corollary we have
$\det(E_1 \cdots E_k) \neq 0$ so
$\det(A) = 0 \iff \det(B) = 0$

But since a matrix with a row of zeroes has determinant equal to zero we see

$$\det(B) = \begin{cases} \det(I) = 1, & \text{if } A \text{ is invertible} \\ 0, & \text{if } A \text{ is not invertible} \end{cases}$$

$\square$

<u>Theorem 2</u>
Let A and B by n by matrices.
Then
$$\det(AB) = \det(A)\det(B)$$

<u>Proof</u>
<u>Case 1</u> A is invertible. Then A is a product of elementary matrices
$A = E_1 E_2 \cdots E_k$ (see the beginning of the proof on the previous page)

Then

$$\det(AB) = \det(E_1 \cdots E_k B)$$
$$= \det(E_1 \cdots E_k)\det(B)$$
$$= \det(A)\det(B).$$

<u>Case 2</u> A is not invertible so $\det(A) = 0$.
So it suffices to show $\det(AB) = 0$
We can write $A = E_1 \cdots E_k C$ where C has bottom row zero. Also
$\det(E_1 \cdots E_k C B) = \det(E_1 \cdots E_k)\det(CB)$
But since C has bottom row zero so does CB. Hence $\det(CB) = 0$ $\square$

<u>Corollary</u>

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

<u>Proof</u> $AA^{-1} = I$

so $\det(AA^{-1}) = 1$

so $\det(A)\det(A^{-1}) = 1$ $\square$

Expansion by Minors

We now give two kinds of formulas for computing determinants (text page 150).

Theorem

(1) Expansion by minors on the j^{th} column:

$$\det A = (-1)^{j+1} a_{1j} \det A_{1j} + (-1)^{j+2} a_{2j} \det A_{2j} + \cdots + (-1)^{j+n} a_{nj} \det A_{nj}$$

(2) Expansion by minors on the i^{th} row:

$$\det A = (-1)^{i+1} a_{i1} \det A_{i1} + (-1)^{i+2} a_{i2} \det A_{i2} + \cdots + (-1)^{i+n} a_{in} \det A_{in}$$

In these formulas A_{ij} is the matrix with the i^{th} row and j^{th} column deleted. The terms $(-1)^{i+j}$ provide alternating signs depending on the position (i,j) in the matrix. The signs can be read off the following:

$$\begin{pmatrix} + & - & + & - & \cdots \\ - & + & - & \cdots & \\ + & - & + & & \\ \vdots & & & \ddots & \end{pmatrix}$$

We now give a formula for the inverse of a matrix A assuming $\det(A) \neq 0$. (text pg 151)

We first define the n by n matrix of <u>signed cofactors</u> $C = C(A) = (c_{ij})$

$$c_{ij} = (-1)^{i+j} \det(A_{ij})$$

We then define the <u>classical adjoint</u> or <u>adjugate</u> of A, denoted $\operatorname{adj}(A)$ by

$$\operatorname{adj}(A) = {}^{t}C = \text{the transpose of } C$$

(see the text pg 146)

In the text $\operatorname{adj}(A)$ is denoted A^*. This is bad notation because we will need the notation A^* later.

We then have

Theorem

$$A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$$

Example The 2 by 2 case

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$C(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$\operatorname{adj}(A) = {}^{t}C(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

The Volume Form D on a n-dimensional Vector Space

We conclude by relating what we have done with the text, Chapter 5.
The text start with a function
D from $\underbrace{F^n \times F^n \times \cdots \times F^n}_{n \text{ copies}}$ to F
satisfying the axioms on page 134.
The text calls D a *determinant function*, I will call it a *volume form*
and develop it in more generality.

So let V be an n-dimensional vector space. Choose a basis
$\mathcal{B} = \{b_1, \ldots, b_n\}$ for V. Then a volume form D on V is a
function of n-tuples of vectors (n-hypervectors) satisfying (see
$D(v_1, v_2, \ldots, v_n)$
Lecture 14, pg 2)

(i) $D(b_1, \ldots, b_n) = 1$.

(ii) $D(v_1, v_2, \ldots, v_n)$ is separately linear in each v_i when the other v_j's are left fixed.

(iii) If v_i and v_j are interchanged then the sign of D changes.

(iv) If v_i is replaced by $v_i + c v_j$ D does not change.

(v) If v_i is replaced by $c v_i$ then D is multiplied by c.

In the text D is constructed and used to construct the determinant. We work in reverse. Let $\mathcal{V} = (v_1, \ldots, v_n)$. Then define

$$D(v_1, \ldots, v_n) = \det\left(\underset{\mathcal{B} \leftarrow \mathcal{V}}{P} \right)$$

<u>Theorem</u> This definition satisfies (i) – (v)

To explain what this means the matrix

$$A = P_{\mathcal{B} \leftarrow \mathcal{V}} \quad \text{is given by}$$

$$A = (a_{ij})$$

where

$$v_j = \sum_{i=1}^{n} a_{ij}\, b_i \, , \quad 1 \leq j \leq n. \qquad (*)$$

In terms of Lecture 13 we have the equality of hypervectors

$$(v_1, \dots, v_n) = (b_1, b_2, \dots, b_n) \cdot A \qquad (**)$$

Then $D(v_1, v_2, \dots, v_n) = \det(A)$.

In the above $\mathcal{V}$ does not have to be a basis. For example we could take $V = \mathbb{R}^2$, $\mathcal{B} = \mathcal{E} = \{e_1, e_2\}$ and $\mathcal{V} = (e_1 + e_2, e_1 + e_2)$. Then

$$P_{\mathcal{B} \leftarrow \mathcal{V}} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ and } \det(\mathcal{V}) = 0$$

Some Examples

$V = \mathbb{R}^2$, $\mathcal{B} = \{e_1, e_2\}$

Let $v_1 = (a, b)$ and $v_2 = (c, d)$

Then

$$(v_1, v_2) = (ae_1 + be_2, ce_1 + de_2) = (e_1, e_2)\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

Hence

$$D(v_1, v_2) = \det\begin{pmatrix} a & c \\ b & d \end{pmatrix} = ad - bc$$

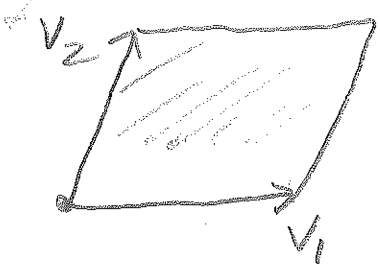
Here is the rule for $D(v_1, v_2, \ldots, v_n)$ for $v_1, v_2, \ldots, v_n \in \mathbb{R}^n$ and $\mathcal{B} = \{e_1, e_2, \ldots, e_n\}$

$$\text{Put } A = \begin{array}{ccccc} v_1 & v_2 & & v_n \\ \left(\begin{matrix} \big\downarrow \\ \end{matrix}\right. & \big\downarrow & \cdots & \left.\begin{matrix} \big\downarrow \\ \end{matrix}\right) \end{array}$$

so A is the matrix whose c

We always take the standard basis for $\mathcal{B}$ in case $V = \mathbb{R}^n$

In fact $|D(v_1, \dots, v_n)|$ is
the volume of the solid in $\mathbb{R}^n$
which is the set of all combinations
$\{t_1 v_1 + \dots + t_n v_n : 0 \le t_i \le 1, 1 \le i \le n\}$.
This is the generalized parallelepiped
spanned by $v_1, v_2, \dots, v_n$:



The case of $n=2$