Lecture 15: Permutations

Permutations

The properties of permutations are discussed in the text, Chapter 9, page 156-160. The notion of the sign of a permutation is closely linked to that of the determinant of a matrix. The set of permutaions of the set $\{1, 2, \ldots, n\}$ forms a group usually denoted Σ_n.

We will first discuss the permutations of any set X.

Definition

Let X be any set. Then the group of permutations of X, denoted $\Sigma(X)$, is the set of bijective (i.e., one-to-one and onto) mappings from X to itself.

$\Sigma(X)$ comes with a noncommutative associative binary opertation, namely composition

$$(f,\, g) \longrightarrow f \circ g.$$

There is a unit element, the identity map $I = I_X$, and every element f has an inverse for the operation $\circ$, namely the inverse mapping f^{-1} to f, that is

$$f \circ f^{-1} = f^{-1} \circ f = I.$$

We will henceforth take $X = \{1,\, 2,\, \ldots,\, n\}$ and abbreviate $\Sigma\,\{1,\, 2,\, \ldots,\, n\}$ to Σ_n.

A permutation σ will often be described by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \ldots & n \\ j_1 & j_2 & j_3 & \ldots & j_n \end{pmatrix}$$

where $\sigma(1) = j_1$, $\sigma(2) = j_2$, …, $\sigma(n) = j_n$.

So

$$\Sigma_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

$$\begin{aligned} \Sigma_3 \quad = \quad & \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \right. \\ & \left. \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\} \end{aligned}$$

Definition

A transposition is a permutation that fixes all but two elements of $\{1, 2, \ldots, n\}$ and interchanges the remaining two elements.

We let τ_{ij} or (ij) be the transposition that interchanges i and j. Note that

$$\tau_{ij}^{1} = \tau_{ij} \cdot \tau_{ij} = I.$$

We say that τ_{ij} has order 2. We will say that I is also a transposition.

Multiplying Permutation

There is a tricky point. We define $\sigma \cdot \sigma_2 = \sigma_1 \circ \sigma_2$ so first do σ_2, then do σ_1:

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \text{ and } \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Let's compute $\sigma_1 \circ \sigma_2$ and $\sigma_2 \circ \sigma_1$. By definition (text, page 159),

$$\sigma_1 \circ \sigma_2(\alpha) = \sigma_1(\sigma_2(\alpha))$$

Multiplying Permutation

So

$$\begin{aligned}
\sigma_1 \circ \sigma_2(1) &= \sigma_1(\sigma_2(1)) = \sigma(1) = 2 \\
\sigma_1 \circ \sigma_2(2) &= \sigma_1(\sigma_2(2)) = \sigma(3) = 3 \\
\sigma_1 \circ \sigma_2(3) &= \sigma_1(\sigma_2(3)) = \sigma(2) = 1
\end{aligned}$$

A better way to do this

$$\left.\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{matrix}\right\} \sigma_2$$

$$\left.\begin{matrix} 2 & 3 & 1 \end{matrix}\right\} \sigma_1$$

The key point: for $\sigma_1 \circ \sigma_2$, you apply σ_2 first.

Permutation Matrices

Let V be a vector space with basis $\{v_1, \ldots, v_n\}$. Then we can map Σ_n into $L(V, V)$ by $\sigma \longrightarrow T)\sigma$ where

$$T_\sigma(v_i) = v_{\sigma(i)}, quad 1 \leq i \leq n$$

Lemma

$\sigma \longrightarrow T)\sigma$ *satisfies*

$$T_{\sigma\tau} = T_\sigma \circ T_\tau, \quad \sigma\tau \in \Sigma_n$$

Proof. For each 1, 2, …, n, we have

$$\begin{aligned} T_\sigma \circ T_\tau(v_i) &= T_\sigma\left(T_\tau(v_i)\right) \\ &= T_\sigma\left(v_{\tau(i)}\right) \\ &= T_{\sigma\left(v_{\tau(i)}\right)} \\ &= T_{\sigma v_{\tau(i)}} \\ &= T_{\sigma\tau}(v_i) \end{aligned}$$

The matrix M_σ of T_σ relative to the basis $\{v_1, \ldots, v_n\}$ has one 1 and $n-1$ zeroes in every row and column.
$n = 3$

$$M_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and

$$M_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Here (123) means the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Lemma

Every permutation is a product of transpositions.

Proof. By inducition on n.

It is true for $n = 1$ (and $n = 2$). Suppose it's true for Σ_n. Let $\sigma \in \Sigma_{n+1}$. Then

$$\sigma(n+1) = i$$

for some $i \in \{1, 2, \ldots, n\}$. Put $\sigma' = \tau_{i,n+1} \cdot \sigma$. Then, $\sigma'(n+1) = n+1$. So we may think of σ' as a element of Σ_n. Hence by induction is a product of transposition (in $\{1, 2, \ldots, n\}$)

$$\sigma' = \prod_{(i,\, j)} \tau_{ij}$$

But (since $\sigma_{i,n+1}^{-1} = \sigma_{i,n+1}$) we have

$$\sigma' = \sigma_{i,n+1} \circ \sigma \Longrightarrow \sigma_{i,n+1} \circ \sigma' = \tau_{i,n+1} \circ \prod_{(i,\, j)} \tau_{ij}.$$

Remark: It is unfortunately true that there are many ways to factor permutation to into transposition.

Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

Then

$$\sigma = (1\,3)(1\,2)$$

and

$$\sigma = (1\,2)(2\,3).$$

This causes problems in proving that $\in(\sigma)$ the sign of a permutation, is well-defined in the next theorem.

Theorem

There exits a unique mapping $\in : \Sigma_n \longrightarrow \{\pm 1\}$ *such that*

1. $\in(\sigma\tau) = \in(\sigma)\in(\tau)$
2. $\in(1) = 1$
3. *If* τ *is a transposition, then*

$$\in(\tau) = -1.$$

Proof. Let σ act on $\mathbb{R}^n$ by permuting the standard basis $\{e_1, \ldots, e_n\}$. Then define

$$\in(\sigma) = \det M_\sigma.$$

This works. □

In many treatments one first proves the existence of $\in(\sigma)$ then for an n by n matrix $A = (a_{ij})$ one defines

$$\det(A) = \sum_{\sigma \in \Sigma_n} a_{1\sigma(1)} a_{n\sigma(1)} \dots a_{n\sigma(1)} \quad (*)$$

In case

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

one gets

$$\det(A) = \underbrace{a_{11}a_{22}}_{\sigma = I} - a_{12}a_{21}$$

The formula we gave in Lecture 13 for the determinant of a 3 by 3 matrix is also a special case of $(*)$.

So roughly defining the sign of a permutation is equivalent to determining the determinant of a matrix.

The definition of $\in$ is unique. Indeed, write σ as a product of k transposition

$$\sigma = \tau_1 \circ \tau_2 \circ \ldots \circ \tau_k.$$

Then

$$\begin{aligned} \in(\sigma) &= \in(\tau_1) \in(\tau_2) \ldots \in(\tau_n) \\ &= (-1)^k \end{aligned}$$

by (3).