Lecture 16

Permutations

The properties of permutations are discussed in the text, §19, pg. 156-160. The notion of the sign of a permutation is closely linked to that of the determinant of a matrix. The set of permutations of the set $\{1, 2, \ldots, n\}$ forms a group usually denoted Σ_n. We will first discuss the permutations of any set X.

Definition

Let X be any set. Then the group of permutations of X, denoted $\Sigma(X)$, is the set of bijective (i.e. one-to-one and onto) mappings from X to itself.

$\Sigma(X)$ comes with a noncommutative associative binary operation, namely composition

$$(f, g) \longrightarrow f \circ g.$$

There is a unit element, the identity map $I = I_X$, and every element f has an inverse for the operation $\circ$, namely the inverse mapping f^{-1} to f.

That is

$$f \circ f^{-1} = f^{-1} \circ f = I$$

We will henceforth ~~take~~ and abbreviate
$X = \{1, 2, \ldots, n\}$ and
$\Sigma(\{1, 2, \ldots, n\})$ to Σ_n

A permutation σ will often be described by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ j_1 & j_2 & j_3 & & j_n \end{pmatrix}$$

where $\sigma(1) = j_1$, $\sigma(2) = j_2, \ldots, \sigma(n) = j_n$

So

$$\Sigma_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}$$

$$\Sigma_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}$$

Definition

A transposition is a permutation that fixes all but two elements of the $\{1, 2, \ldots, n\}$ and interchanges the remaining two elements

We let τ_{ij} or (ij) be the transposition that interchanges i and j.

Note that $\tau_{ij}^2 = \tau_{ij} \cdot \tau_{ij} = I$ (we say τ_{ij} has order two).

We will say that I is also a transposition.

Multiplying Permutation

There is a tricky point here: We define $\sigma_1 \bullet \sigma_2 = \sigma_1 \circ \sigma_2$ so first do σ_2 then do σ_1

$$\sigma_1 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \ ; \ \sigma_2 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

Let's compute $\sigma_1 \circ \sigma_2$ and $\sigma_2 \circ \sigma_1$

By definition (text pg 158)

$$\sigma_1 \circ \sigma_2 (x) = \sigma_1(\sigma_2(x))$$

So
$$\sigma_1 \circ \sigma_2 (1) = \sigma_1(\sigma_2(1)) = \sigma_1(1) = 2$$
$$\sigma_1 \circ \sigma_2 (2) = \sigma_1(\sigma_2(2)) = \sigma_1(3) = 3$$
$$\sigma_1 \circ \sigma_2 (3) = \sigma_1(\sigma_2(3)) = \sigma_1(2) = 1$$

A better way to do this

$$\begin{matrix} 1 & 2 & 3 \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{matrix} \quad \begin{matrix} \big\} \sigma_2 \\ \big\} \sigma_1 \end{matrix}$$

The key-point: for $\sigma_1 \circ \sigma_2$ you apply σ_2 first.

Permutation Matrices

Let V be a vector space with basis $(v_1, v_2, \ldots, v_n)$. Then we can map Σ_n into $L(V, V)$ by

$\sigma \longrightarrow T_\sigma$ where

$$T_\sigma(v_i) = v_{\sigma(i)}, \quad 1 \le i \le n$$

Lemma

$\sigma \longrightarrow T_\sigma$ satisfies

$$T_{\sigma\tau} = T_\sigma \circ T_\tau, \quad \sigma, \tau \in \Sigma_n$$

<u>Proof</u> For each $i = 1, 2, \ldots, n$ we have

$$\begin{aligned}
(T_\sigma \circ T_\tau)(v_i) &= T_\sigma(T_\tau(v_i)) \\
&= T_\sigma(v_{\tau(i)}) \\
&= v_{\sigma(\tau(i))} \\
&= v_{\sigma\tau(i)} \\
&= T_{\sigma\tau}(v_i)
\end{aligned}$$

$\square$

The matrix M_σ of T_σ relative to the basis $\{v_1, v_2, \ldots, v_n\}$ has one 1 and $n-1$ zeroes in every row and column.

$n = 3$

$$M_{(12)} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $$M_{(123)} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Here (123) means the permutation

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Lemma

Every permutation is a product of transpositions

Proof

By induction on n. It is true for $n = 1$ (and $n=2$). Suppose it is true for Σ_n. Let $\sigma \in \Sigma_{n+1}$.

Then $\sigma(n+1) = i$ some $i \in \{1, 2, \ldots, n\}$

Put $\sigma' = \tau_{i,n+1} \circ \sigma$.

Then $\sigma'(n+1) = \tau_{i,n+1}(\sigma(n+1)) = \tau_{i,n+1}(i) = n+1$

Hence $\sigma'(n+1) = n+1$. So we may think of σ' as an element of Σ_n.

Hence by induction is a product of transpositions (in $\{1, 2, \ldots, n\}$)

$$\sigma' = \prod_{(i,j)} \tau_{ij}.$$

But (since $\tau_{i,n+1}^{-1} = \tau_{i,n+1}$)
we have

$$\sigma' = \tau_{i,n+1} \circ \sigma \Rightarrow \sigma = \tau_{i,n+1} \circ \sigma' = \tau_{i,n+1} \circ \prod_{i,j} \tau_{i,j}$$

$\square$

<u>Remark</u>

It is unfortunately true that there are many ways to factor a permutation into transpositions

Consider the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$$

Then $\sigma = (13)(12)$

and $\sigma = (12)(23)$

This cause problems in proving $\epsilon(\sigma)$, the sign of a permutation, is well-defined in the next theorem.

The Sign of a Permutation

Theorem

There exists a unique mapping $\epsilon : \Sigma_n \longrightarrow \{\pm 1\}$ such that

(1) $\epsilon(\sigma\tau) = \epsilon(\sigma)\,\epsilon(\tau)$

(2) $\epsilon(1) = +1$

(3) If τ is a transposition then $\epsilon(\tau) = -1$

Proof

Let σ act on $\mathbb{R}^n$ by permuting the standard basis $\{e_1, e_2, \ldots, e_n\}$. Then define

$$\epsilon(\sigma) = \det(M_\sigma)$$

This works $\square$

In many treatments one first proves the existence of $\epsilon(\sigma)$ then for an n by n matrix $A = (a_{ij})$ one defines

$$\det(A) = \sum_{\sigma \in \Sigma_n} \epsilon(\sigma)\, a_{1\sigma(1)}\, a_{2\sigma(2)} \cdots a_{n\sigma(n)} \quad (*)$$

In case

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

one gets

$$\det(A) = \underbrace{a_{11}\, a_{22}}_{\sigma = I} - \underbrace{a_{12}\, a_{21}}_{\sigma = (12)}$$

The formula we gave in Lecture 13 for the determinant of a 3 by 3 matrix is also a special case of $(*)$

So roughly defining the sign of a permutation is equivalent to defining the determinant of a matrix.

The definition of ϵ is unique. Indeed, write σ as a product of k transpositions

$$\sigma = \tau_1 \circ \tau_2 \circ \cdots \circ \tau_k$$

Then
$$\epsilon(\sigma) = \epsilon(\tau_1)\,\epsilon(\tau_2) \cdots \epsilon(\tau_k)$$
$$= (-1)^k \quad \text{by (3)}$$