Selected HW2 Solutions

1. 3.11, 3)
$$TU(a,b) = T(U(a,b)) = T(a+b,a) = (-2a-3b,b)$$
whereas
$$UT(a,b) = U(T(a,b)) = U(-3a+b,a-b) = (-2a,-3a+b).$$
So $UT \neq TU$.

2. 3.11, 9) We will prove $(a) \iff (b)$ and then $(a)+(b) \iff (c)$ and $(a)+(b) \iff (d)$. The main point is proving $(a) \iff (b)$ since the other two are almost by definition.

   Suppose T is one-to-one and V has dimension n. Since T is one-to-one, $N(T) = \{0\}$ and thus $dimN(t) = 0$. Thus, by the rank-nullity theorem, $dim(T(V)) = n$ and since V is n-dimensional we have that $T(V) = V$, hence T is onto.

   Suppose now that T is onto. Then $T(V) = V$ and hence $dim(T(V)) = n$. Now by the rank-nullity theorem, $dimN(t) = 0$ and hence T is one-to-one.

   We have seen that being one-to-one and onto is equivalent to being an isomorphism, hence $(a)+(b) \iff (c)$.

   We have also proven that being one-to-one and onto is equivalent to being invertible, hence $(a)+(b) \iff (d)$.

3. 3.13, 4) We want to show that $dim(T(V)) = rank(A)$. Fix a basis $\mathcal{B} = (b_1, \ldots, b_n)$. Then we know $T(V) = S(T(b_1), \ldots, T(b_n))$, hence
$$dimT(V) = dimS(T(b_1), \ldots, T(b_n)).$$
On the other hand, by definition, since rank(A) is the dimension of the column space and the columns of A are $[T(b_1)]_{\mathcal{B}}, \ldots, [T(b_n)]_{\mathcal{B}}$, we have
$$rank(A) = dimS([T(b_1)]_{\mathcal{B}}, \ldots, [T(b_n)]_{\mathcal{B}}).$$
But, the map "sub-$\mathcal{B}$" (choosing coordinates) is an isomorphism, hence these dimensions are the same.