LECTURE 3

Probability Computations

Pg. 67, #42 is one of the hardest problems in the course. The answer is a simple fraction so there should be a simple way to do it. I don't know a simple way – I used the formula for $P(A \cup B \cup C)$, text pg.56 or Lecture 1, pg. 30. If you find a simple way show it to me or your TA . We will remember it when it comes to final grade time. The rest of this lecture will be about *Bridge Hands and Poker Hands.*

Bridge Hands

If you play bridge you get dealt a hand of 13 cards. Let S be the set of all bridge hands (so our "experiment" is dealing 13 cards).

Since the order in which you receive the cards doesn't count (it never does in card games)

$$\begin{aligned} \#(S) &= \binom{52}{13} \\ &= \text{the number of 13 element subsets of a 52 element set.} \end{aligned}$$

We will now compute the probability of certain bridge hands.

1. Let $A =$ the hand is all hearts What is $\#(A)$? We use the "principle of restricted choice" (I made up this name, it isn't in common usage). Our choice is *restricted* to the subset of hearts – we have to choose 13. There are 13 hearts so we have $\binom{13}{13} = 1$ hands. So

$$P(A) = \frac{\#(A)}{\#(S)} = \frac{1}{\binom{52}{13}}$$

(A is very unlikely).

2. Let

$$B = \text{there are no hearts}$$

What is $\#(B)$? Once again, we use the "principle of restricted choice". We have to choose 13 nonhearts. There are $52 - 13 = 39$ nonhearts so we have to choose 13 things from 39 things so

$$\#(B) = \binom{39}{13}$$

So

$$P(B) = \frac{\binom{39}{13}}{\binom{52}{13}}$$

Poker Hands

If you play poker you get dealt a hand of 5 cards (or sometimes 7 cards). For the next few examples we will assume we are playing "5 -card poker". There is also the role of aces. In the text pg. 67, #43 aces can be "either high or low". This means that you can count them as 1's (i.e., lower than anything else) or higher than anything else so we have in order

$$\underset{\text{ace}}{\text{A}} \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; 8 \;\; 9 \;\; 10 \;\; \underset{\text{jack}}{\text{J}} \;\; \underset{\text{queen}}{\text{Q}} \;\; \underset{\text{king}}{\text{K}} \;\; \underset{\text{ace}}{\text{A}}.$$

There are $\begin{pmatrix} 52 \\ 5 \end{pmatrix}$ (five-card) poker hands.

1. **A "Straight"**

A = all five cards are consecutive = a "straight",
e.g., 5 6 7 8 9
or A 2 3 4 5 (aces are low)
or 10 J Q K A (aces are high).

Find $P(A)$, this is Problem 43 from the text.

There is one observation you need – a *straight is determined by its bottom (i.e., lowest) card.* You need one more observation: the cards J, Q, K cannot be bottom cards for a straight (because there aren't enough cards above them).

The last straight is 10 J K Q A ← aces are high. Also A can be a bottom card because aces are low A 2 3 4 5. So there are $13 - 3 = 10$ bottom cards so there are 10 different kinds of straights – so at first glance one might think there are 10 straights. BUT, let's consider A 2 3 4 5. We haven't taken account of the SUITS of the cards. The A could be any one of four suits, for each these the 2 could be any one of four suits so there are 4^5 straights of the form A 2 3 4 5. Hence,

$$\#(A) = \underbrace{(10)}_{\substack{\text{lowest}\\\text{card}}} \;\; \underbrace{(4^5)}_{\substack{\text{suit}\\\text{of}\\\text{each}\\\text{card}}}$$

so

$$P(A) = \frac{(10)(4^5)}{\binom{52}{5}}.$$

2. **A "Flush"**

A "flush" is a poker hand in which all cards are the same suit (spade, heart, diamond and club).

Let B = set of all flushes.

First, how many hands are there that are all hearts?

We use the principle of restricted choice. We choose five cards but our choice is restricted to hearts. There are 13 hearts, we have to choose 5. So there are $\binom{13}{5}$ such hands and we have

$$\#(B) = \underbrace{(4)}_{\substack{\text{choose} \\ \text{a suit}}}$$

So

$$P(B) = \frac{(4)(\binom{13}{5})}{\binom{52}{5}}.$$

3. **A "Straight Flush"**

Lets combine 2. and 3. so C = set of all straights so that all the cards have the same suite. To compute $\#(C)$ procede as follows.

1. Pick the lowest card - 10 ways.

2. Pick the common suit that all five cards have - 4 ways.

We obtain
So

$$\#(C) = (10)(4) = 40$$

and

$$P(C) = \frac{(40)}{\begin{pmatrix} 52 \\ 5 \end{pmatrix}} = \begin{matrix} \text{a} \\ \text{very small} \\ \text{number} \end{matrix}.$$

4. **A "Full House"**

A full house is a poker hand which consists of 3 of one kind and 2 of a different kind, e.g., JJJ KK. Let D = set of full houses. Here is how we compute $\#(D)$:

1. Pick an *ordered* pair of kinds, e.g., J or K.

2. Pick 3 of the first and 2 of the second.

so in the above case we get JJJ KK.

Key test to perform

Were we right when we said ORDERED pair above? So let's test, reverse the order K, J and check if we get a different hand. If we do we were right to say ORDERED pairs. If we don't we were wrong and we have to replace ORDERED pairs in 1. by UNORDERED pairs. Okay, if we take the pair K, J and do 2. we get the hand JJJ KK.

Are KKK JJ and JJJ KK different? *Yes*, the first beats the second so we were right to pick ordered pairs. Now lets finish the job

1. There are 13 kinds, on ordered pair of kinds is a 2 permutation of the 13 element set of kinds, so

$$\#(1.) = P_{2,13} = \frac{(13)!}{(11)!} = (13)(12)$$

2. Now there are $\binom{4}{3}$ to pick the first kind (say the first kind is a jack so we have to pick 3 of the 4 jacks) and $\binom{4}{2}$ ways to pick the second kind so

$$\#(D) = \underbrace{(13)(12)}_{\text{2 kinds}}\binom{4}{3}\binom{4}{2}$$

so

$$P(D) = \frac{(13)(12)\binom{4}{3}\binom{4}{2}}{\binom{52}{5}}.$$

5. **"Two Pair"**

In poker "two pair" is a hand which consists of 2 of one kind, 2 of a second (*different*) kind and one more of a third kind (*different* from the other two), e.g., JJ KK 10.

Let E be the set of two-pair hands. Here is how we compute $\#(E)$.

1. Pick an $\left\{\begin{array}{l}\text{ordered}\\ \text{unordered}\end{array}\right.$ pair of kinds, e.g., J,K.
2. Pick a third kind.
3. Pick two each of the first two kinds and one of the third kind.

$64,000 Question

In 1. do we pick on *ordered* pair or an *unordered* pair?

Answer

Do the test?

The pair J,K $\longrightarrow$ JJKK

The pair K,J $\longrightarrow$ KKJJ

In poker JJKK and KKJJ are the *SAME* so J,K and K,J give the *SAME* and so order does not matter, so we pick on *unordered* pair.

Now we compute $\#(E)$.

1. There are 13 kinds. We want the number of *unordered* pairs. It is

$$\binom{13}{2} = \frac{(13)(12)}{2} = \frac{1}{2}.$$

   This is the number we got for 1. in the full house case.

2. We have chosen two kinds. There are $13 - 2 = 11$ left. We have to pick one so

$$\binom{11}{1} = 11.$$

3. Now we have to choose the cards (suits) in each of the three above kinds. There are four cards of each kind. We have to choose two cards of the first kind, this we can do in $\binom{4}{2} = 6$ ways and the same for the second kind (there is no first or second, but it does matter the numbers are the same). We have to choose 1 card of the third kind so there are $\binom{4}{1} = 4$ choices for the third kind. Hence:

$$\#(E) = \left(\frac{(13)(12)}{2}\right)(11)\binom{4}{2}\binom{4}{2}\binom{4}{1} = (78)(11)(6)(6)(4)$$

   so

$$P(D) = \frac{(78)(11)(6)(6)(4)}{\binom{52}{5}}$$