Stat 400, Midterm 2 (Fall 2002)

Solutions

1.

$x \backslash y$	0	1	
0	0	$\frac{1}{4}$	$\frac{1}{4}$
1	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$
	$\frac{1}{4}$	$\frac{3}{4}$	

(a)

x	0	
$P(X=x)$	$\frac{1}{4}$	$\frac{3}{4}$

y	0	1
$P(Y=y)$	$\frac{1}{4}$	$\frac{3}{4}$

(b) No, since the upper left entry in the matrix is 0 instead of $\left(\frac{1}{4}\right)\left(\frac{1}{4}\right) = \frac{1}{16}$

:) Z takes values 0, 1, 2

z	0	1	2
$P(Z=z)$	0	$\frac{1}{2}$	$\frac{1}{2}$

) We first compute

$$E(XY) = (1)(1)\left(\tfrac{1}{2}\right) = \frac{1}{2}$$

Then $E(X) = E(Y) = (1)\left(\frac{3}{4}\right) = \frac{3}{4}$

ence

$$\text{ov}(X, Y) = E(XY) - E(X)E(Y)$$

$$= \frac{1}{2} - \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)$$

$$= \frac{8}{16} - \frac{9}{16} = -\frac{1}{16}$$

(e) To find the correlation we need $\sigma_X = \sigma_Y$.

Now $E(X^2) = (1)^2\left(\frac{3}{4}\right) = \frac{3}{4}$

so

$$V(X) = E(X^2) - E(X)^2$$

$$= \frac{3}{4} - \left(\frac{3}{4}\right)^2 = \frac{3}{4} - \frac{9}{16}$$

$$= \frac{12}{16} - \frac{9}{16} = \frac{3}{16}$$

$\sigma_X = \frac{\sqrt{3}}{4}$ and $\sigma_Y = \frac{\sqrt{3}}{4}$.

Hence

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-\frac{1}{16}}{\frac{\sqrt{3}}{4}\,\frac{\sqrt{3}}{4}}$$

$$= -\frac{1}{3}$$

2. In Fall 2002 I showed the students a "machine" for finding the distribution of a sum $Z = X + Y$ when you know the distributions of X and Y. Since you don't have the machine available this goes from a "good citizen problem" to a hard problem.

To do this problem it would be a good idea to have downloaded the "distribution" file from the end of my web page BACKGROUND EXTRA COURSE NOTES

The solution of the problem consists of four steps

(i) <u>Look up the pmf of X and Y</u>

The geometric distribution is the fifth item on "Discrete Distributions" on the above file. Otherwise you have to go to page 126 of the text (middle of the page) to find

$$p_X(x) = nb(x; 1, p) = (1-p)^x p = q^x p$$

$$x = 0, 1, 2, \ldots$$

so $p_Y(y) = q^y p, \; y = 0, 1, 2, \ldots$

(ii) Using the X and Y are independent compute the joint pmf $P_{X,Y}(x,y)$ of X and Y

From (i) we have

$$P_X(x) = P(X=x) = q^x p$$

$$P_Y(y) = P(Y=y) = q^y p$$

so.

$$P_{X,Y}(x,y) = P(X=x, Y=y)$$

by independence

$$= P(X=x)\,P(Y=y)$$

$$= q^x p \, q^y p$$

$$= q^{x+y} p^2 \qquad (*)$$

Computing the joint pmf for Problem 2 of Midterm 2, Fall 2012

$q^{x+y}p^2$

$x \backslash y$	0	1	2		y		∞
0	p^2	qp^2	q^2p^2		$q^y p^2$		p
1	qp^2	q^2p^2	q^3p^2				qp
2	q^2p^2	q^3p^2	q^4p^2		$q^{y+2}p^2$		q^2p
							$\vdots$
x	$q^x p^2$	$q^{x+1}p^2$			$q^{x+y}p^2$		$q^x p$
							$\vdots$
∞	p	qp	q^2p	$\cdots$	$q^y p$	$\cdots$	

The margins: geometric distribution with parameter p

x	0	1	2		x	
$P(X=x)$	p	qp	q^2p	$\cdots$	$q^x p$	$\cdots$

the same distribution

y	0	1	2		y	
$P(Y=y)$	p	qp	q^2p	$\cdots$	$q^y p$	$\cdots$

(iii) Using the formula (*) for the joint pmf $P_{X,Y}(x,y)$ compute the pmf $P_Z(z)$ for the sum

$$P_Z(z) = P(X+Y=z) = \sum_{x=0}^{z} P(X=x, Y=z-x)$$

from (*)

$$= \sum_{x=0}^{z} q^{x+z-x} p^2 = \sum_{x=0}^{z} q^z p^2$$

The expression inside the sum does not depend on x so we are summing the same thing from $x = 0$ to z ($z+1$ terms) so

$$P_Z(z) = (z+1)\, p^2 q^z \qquad (**)$$

iv) Inverting step (i) — the recognition problem

Now you have to look over all the discrete distributions we have studied to find the formula (**) - This is where you absolutely have to have printed out the distribution file so you have only to scan one page (the sum of discrete random variables is discrete)
The required distribution is the fourth one on the list (the negative binomial with $r=2$)

$$P(X=k) = \binom{k+r-1}{r-1} p^r q^k$$

$$\overset{r=2}{=} \binom{k+1}{1} p^2 q^k$$

$$= (k+1) p^2 q^k \quad \left(\text{put } k=z \atop \text{that's } (**)\right)$$

4 ; ;We are compute $T_{X,Y}$ 10

(a) $P(X=0 \mid Y=0) = \dfrac{P(X=0, Y=0)}{P(Y=0)}$

$= \dfrac{0}{\frac{1}{4}} = 0$

$P(X=0 \mid Y=1) = \dfrac{P(X=0, Y=1)}{P(Y=1)}$

$= \dfrac{\frac{1}{4}}{\frac{3}{4}} = \dfrac{1}{3}$

$P(X=1 \mid Y=0) = \dfrac{P(X=1, Y=0)}{P(Y=0)}$

$= \dfrac{\frac{1}{4}}{\frac{1}{4}} = 1$

$$P(X=1 \mid Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)}$$

$$= \frac{\frac{1}{2}}{\frac{3}{4}} = \frac{2}{3}$$

(11)

$x \backslash y$	0	1	
0	0	$\frac{1}{3}$	
1	1	$\frac{2}{3}$	

Remark

The first column is the probability mass function for a discrete random variable denoted $X|0$.

The second column is the probability mass function for a discrete random variable denoted $X|1$.

4. Since $X \sim U(0,1)$

we have

$$f_X(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

We have $y = h(x) = \sqrt{x}$. This

on the support of f_X

function is strictly increasing

so we may apply the Engineer's Way

note that $h(0) = 0$, $h(1) = 1$ so the support of Y is still $[0,1]$)

The inverse function to $y = \sqrt{x}$ is $x = y^2$. So putting $g(y) = y^2$ we have for $0 \le y \le 1$,

$$f_Y(y)\,dy = f_X(g(y))\,d(g(y)) = f_X(y^2)\,dy^2$$

$$= 1\,d(y^2) = 2y\,dy$$

So $f_Y(y) = 2y$ for $0 \le y \le 1$ and 0 otherwise