Lecture 7

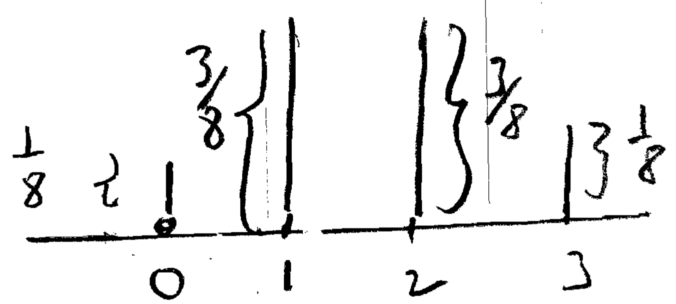
Change of discrete random variable

You have already seen (I hope) that whenever you have "variables" you need to consider change of variables. Random variables are no different.

The notion of "change of random variable" is handle too briefly on page 103 of the text. This is something I will test you on.

Example 1 Suppose $X \sim Bin(3, \frac{1}{2})$

line graph



table

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b)

Now we have computed the possible values of Y we need to compute their probabilities. Just repeat what we did

$$P(Y=-1) = P(2X-1=-1)$$
$$= P(X=0) = \frac{1}{8}$$

$$P(Y=1) = P(2X-1=1)$$
$$= P(X=1) = \frac{3}{8}$$

Similarly

$P(Y=3) = \frac{3}{8}$ and $P(Y=5) = \frac{1}{8}$

X	-1	1	3	5
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Suppose want to define a new random variable $Y = 2X - 1$.

How do we do it?

So how do we define $P(Y=k)$?

Answer - express Y in terms of X and compute so

$$P(Y=k) = P(2X-1=k)$$

$$= P\left(X = \frac{k+1}{2}\right) \quad (*)$$

The right-hand side is the logical definition of the left-hand side.

but as is often the case in probability it is easier to pretend we know what $P(Y=k)$ means already and then the next two steps are a computation

So let's compute the pmf of Y.

What are the possible values of Y?

From (*) k is a possible value of $Y \iff \frac{k+1}{2}$ is a possible value of X

$$\iff \frac{k+1}{2} = \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \iff Y = \begin{cases} -1 \\ 1 \\ 3 \\ 5 \end{cases}$$

Note $\frac{k+1}{2} = x \iff k = 2x - 1$

possible value of X

possible value of Y

So the possible values of Y are obtained by applying the function $h(x) = 2x - 1$ to the possible values of X (note $Y = f(X)$).

$$x \xrightarrow{h} 2x-1$$

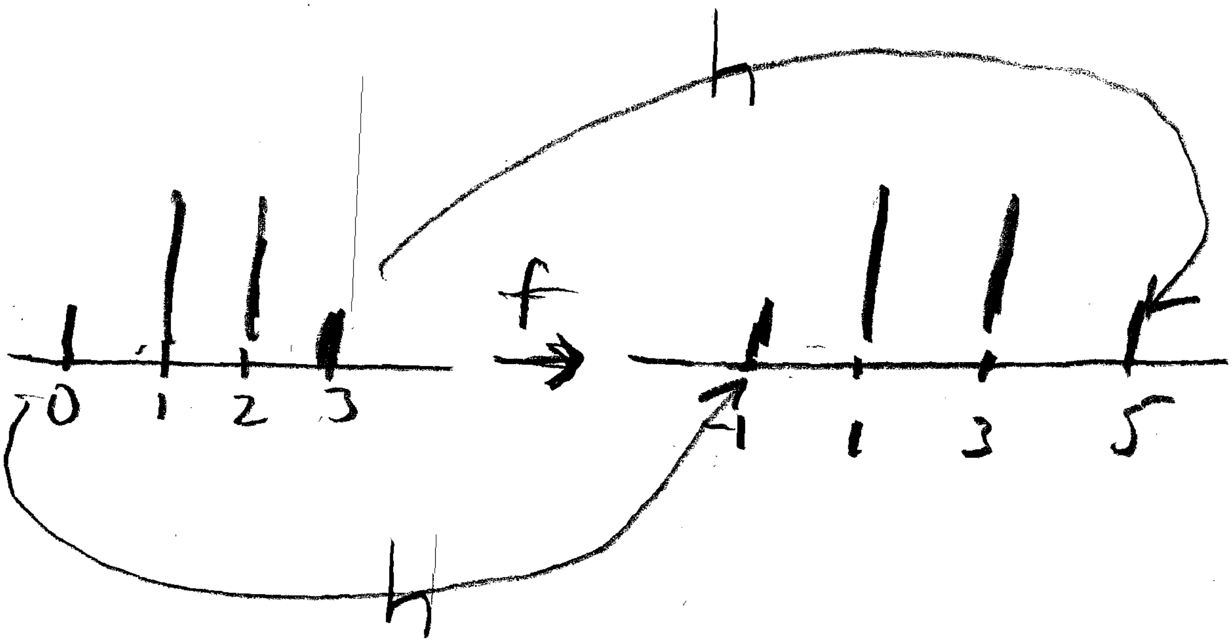
$$0 \longrightarrow -1$$
$$1 \longrightarrow 1$$
$$2 \longrightarrow 3$$
$$3 \longrightarrow 5$$

possible values of X

possible values of Y.

Just "push forward" the values of X.

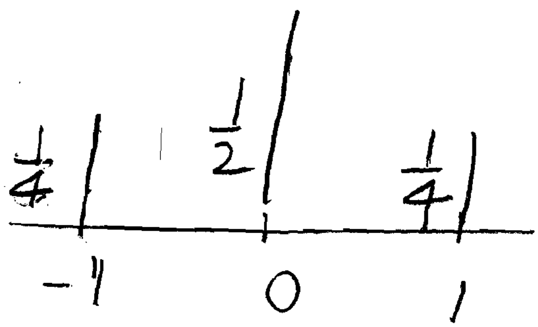
So we have the "same probabilities" as before namely $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ it is just they are pushed-forward to new locations



Example 2 (Probabilities can "coalesce")

There is one tricky point. Several different possible values of X can push-forward to the same values of Y. We now give an example

Suppose X has pmf



Problem (from an old midterm)
Find a linear change of variable $Y = aX + b$ so that $Y \sim Bin(2, \frac{1}{2})$

We will make the change of variable $Y = X^2$.
So what happens when we push forward the <u>three</u> values $-1, 0, 1$ by $h(x) = x^2$.

We get only the <u>two</u> values 0 and 1.

$$\begin{aligned} -1 &\xrightarrow{h(x)} 1 \\ 0 &\longrightarrow 0 \\ 1 &\longrightarrow 1 \end{aligned}$$

What happens with the corresponding probabilities

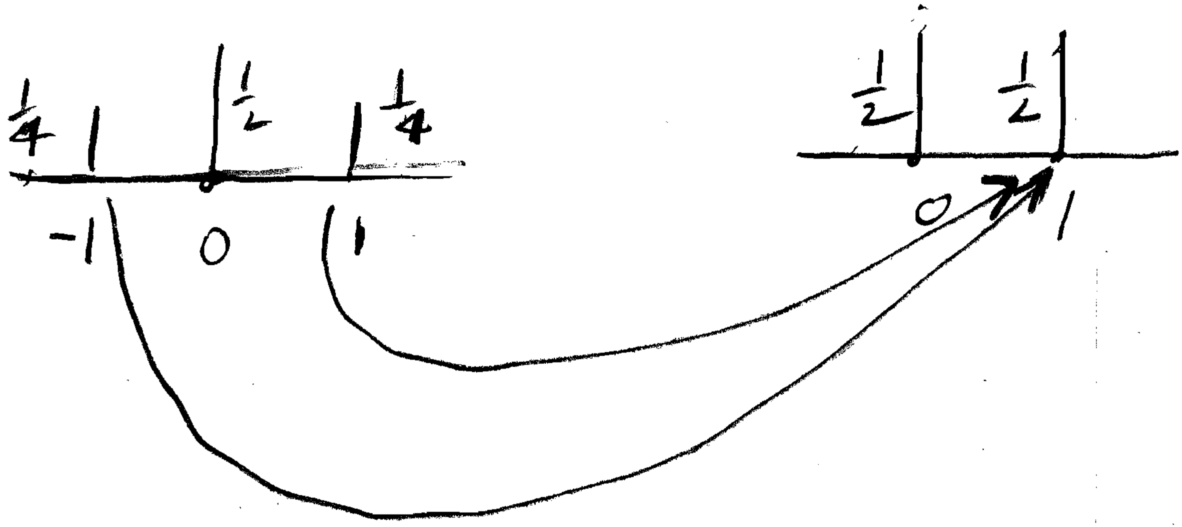
$$P(Y=0) = P(X^2=0) = P(X=0) = \tfrac{1}{2}$$

BUT

$$P(Y=1) = P(X^2=1) = P(X=1 \text{ or } X=-1)$$
$$= P((X=1) \cup (X-1))$$
$$= P(X=1) + P(X=-1)$$
$$= \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

So we set

y	0	1
$P(Y=y)$	$\frac{1}{2}$	$\frac{1}{2}$



Think of two masses (probabilities) of mass $\frac{1}{4}$ coalesing into a combined mass of $\frac{1}{2}$.

The Expected Value Formula

If $h(x)$ in the transformation law $Y = h(X)$ is complicated it can be very hard to explicitly compute the pmf of Y. Amazingly we can compute the expected value $E(Y)$ using the old pmf $p_X(x)$ of X according to

Theorem

$$E(h(X)) = \sum_{\substack{\text{possible} \\ \text{values of } X}} h(x)\, p_X(x)$$

$$= \sum_{\substack{\text{possible values} \\ \text{of } X}} h(x)\, P(X = x)$$

We will illustrate this with the pmf's of Example 1

First we compute $E(Y)$ using the definition of $E(Y)$

y	-1	1	3	5
$P(Y=y)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(#)

$$E(Y) = \sum_{\substack{\text{possible}\\ \text{values}\\ \text{of } Y}} y\, P(Y=y)$$

$$= (-1)\left(\tfrac{1}{8}\right) + (1)\left(\tfrac{3}{8}\right) + (3)\left(\tfrac{3}{8}\right) + (5)\left(\tfrac{1}{8}\right)$$

$$= \frac{-1+3+9+5}{8}$$

$$= \frac{16}{8} = 2$$

Notice to do the previous computation we needed the table (#) which we computed on page 5.

Now we use the Theorem

So now we use that Y is a function of the random variable X and use the pmf of X from the table on page 1

x	0	1	2	3
$P(X=x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b)

$$E(X) = \sum_{\substack{\text{possible}\\ \text{values}\\ \text{of } X}} h(x)\,P(X=x)$$

$$= \sum_{x=0,1,2,3} (2x-1)\,P(X=x)$$

$$= (-1)\left(\tfrac{1}{8}\right) + (1)\left(\tfrac{3}{8}\right) + (3)\left(\tfrac{3}{8}\right) + (5)\left(\tfrac{1}{8}\right) = 2$$

The most common change of variable is linear $Y = aX + b$ so we will give formulas to show how expected value and variance behave under such a change

Theorem

(i) $E(aX+b) = aE(X) + b$

(ii) $V(aX+b) = a^2 V(X)$

(so $V(-X) = V(X)$)