Lecture 12

The Basic Continuous Distributions

We will now study the basic examples

1. The normal distribution.

2. The gamma distribution with special cases

3. The exponential distribution

and

4. The chi-squared distribution.

5. The Student t-distribution.

This is the most important lecture in the course.

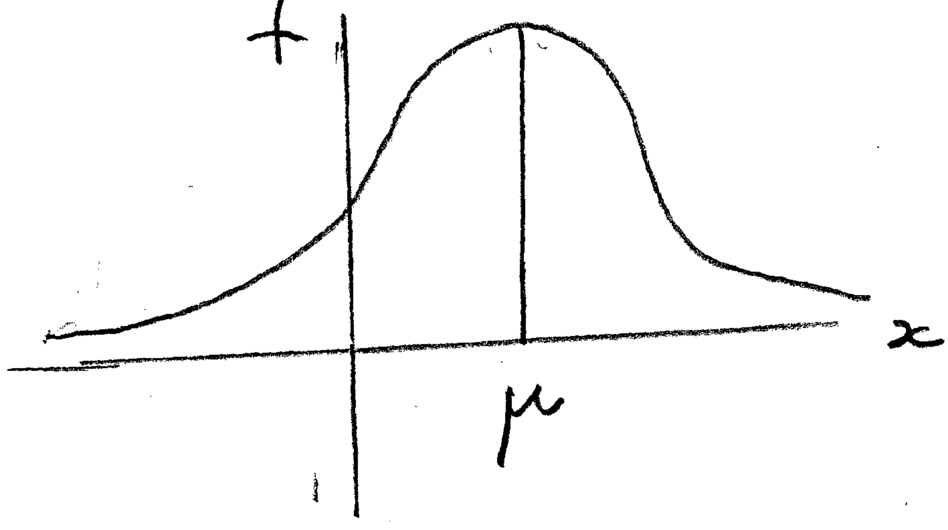
This lecture is all about the most important distribution.

The Normal Distribution

Definition A continuous random variable X has normal distribution with parameters μ and σ^2, denoted $X \sim N(\mu, \sigma^2)$, if the pdf f of X is given by

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad -\infty < x < \infty$$

The graph of f is the "**bell curve**"

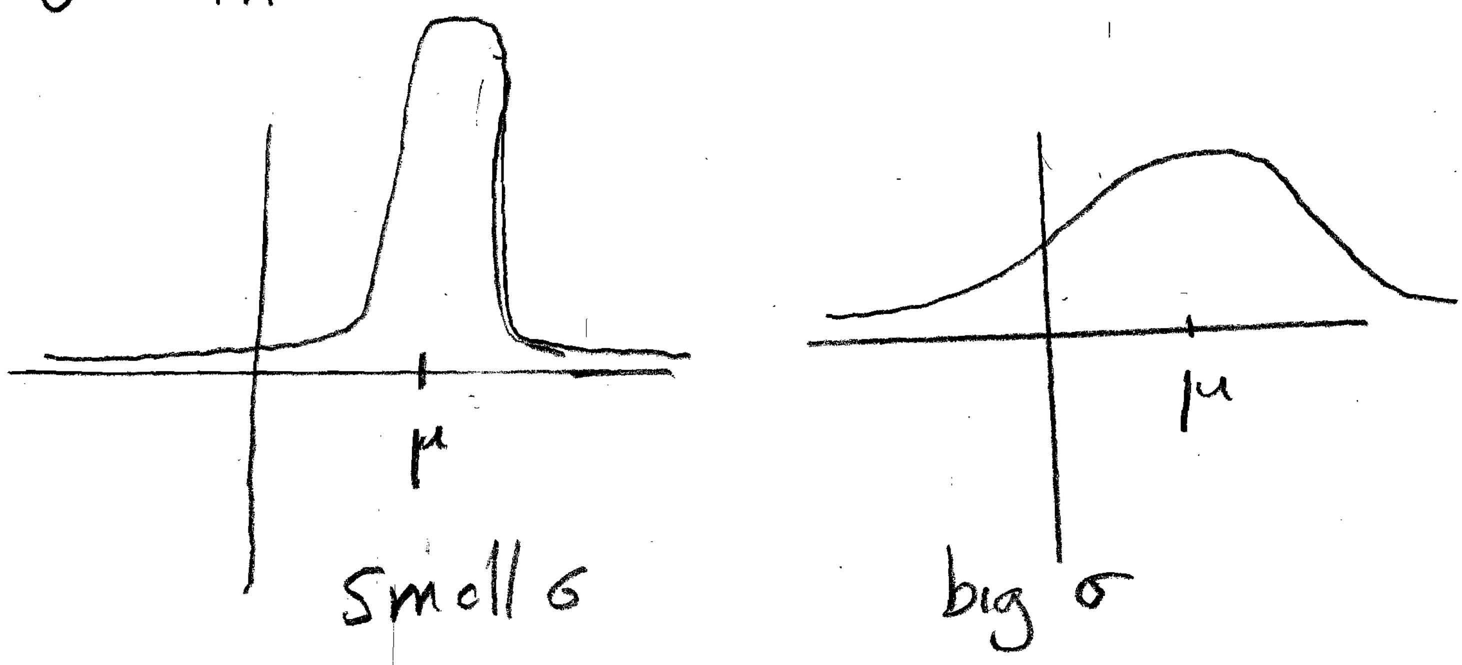


μ is a point of symmetry of f so by Lecture 11, page 15

$$E(X) = \mu.$$

(this why this parameter is called μ).

σ^2 measures the "width" of the curve



<u>Proposition</u> If $X \sim N(\mu, \sigma^2)$ then

(i) $E(X) = \mu$

(ii) $V(X) = \sigma^2$

(this justifies the names of the parameters)

Remark If $X \sim N(\mu, \sigma^2)$ then

$$P(a \leq X \leq b) = \int_a^b \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

This integral cannot be computed by calculus methods so it must be computed by numerical analysis methods. However these probabilities can be recovered from the table in the front flap of the text or from a computer. To do this we need to reduce to the "standard" case $\mu = 0, \sigma = 1$ (otherwise we would need infinitely many tables, one for each pair (μ, σ^2)). The reduction to the standard case is called <u>standardization</u>.

The Standard Normal Distribution

Definition A normal distribution with mean 0 and variance 1 (so $\mu = 0$ and $\sigma^2 = 1$, so $\sigma = 1$) is called a **standard normal distribution**.

A random variable with standard normal distribution will be denoted Z so $Z \sim N(0,1)$.

The pdf $f(z)$ for Z is given by

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} z^2}, \quad -\infty < z < \infty$$

(see the next page for the graph of f)

The function on the right is often called the Gaussian and comes up all over mathematics. It gives rise to the famous theta functions in number theory.

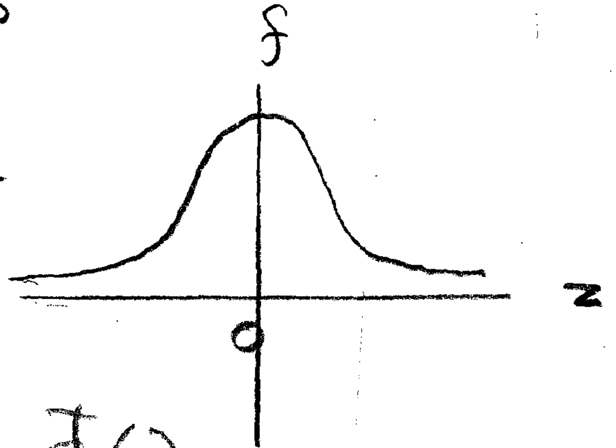
Definition

The cumulative distribution function of the normal distribution will be denoted $\Phi(z)$. So

$$\Phi(z) = P(Z \le z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}\, dt$$

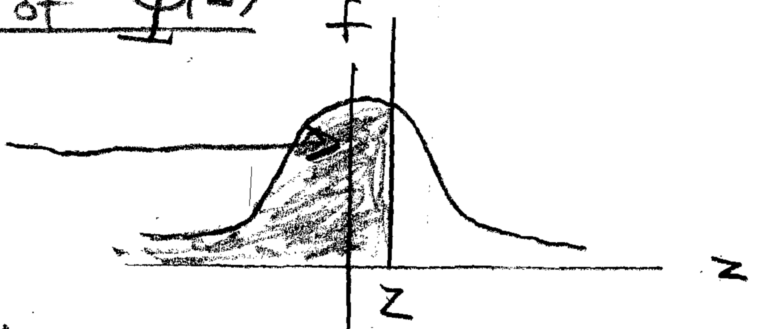
Pictures

graph of f

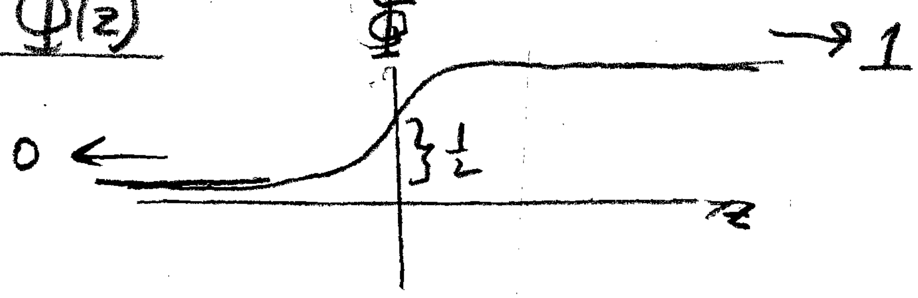


meaning of $\Phi(z)$

this area is $\Phi(z)$



graph of $\Phi(z)$



$$\lim_{z \to -\infty} \Phi(z) = 0$$

$$\lim_{z \to \infty} \Phi(z) = 1$$

Using the tables on page 668-669

The values of $\Phi(z)$ are tabulated in the front flaps of the text or better from the web – see next page.

Problem

From the table in the front flap or the web

(a) Compute $P(Z \leq 1.25)$ (0.8944)

(b) Compute $P(Z \leq -1.25)$

(c) Compute $P(-1.25 \leq Z \leq 1.25)$

The challenge is to use the answer to (a) namely $.8944$ to do (b) and (c). In other words to do all three parts you have to look up only one value.

First we show (a) gives (b).

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Stat Trek Teach yourself statistics

Cumulative Normal Distribution Calculator: Online Statistical Table

The Normal Distribution Calculator makes it easy to compute cumulative probability, given a normal random variable; and vice versa. For help in using the calculator, read the Frequently-Asked Questions or review the Sample Problems.

To learn more about the normal distribution, go to Stat Trek's tutorial on the normal distributi

- Enter a value in three of the four text boxes.
- Leave the fourth text box blank.
- Click the **Calculate** button to compute a value for the blank text box.

Standard score (z)	
Cumulative probability P(Z ≤ z)	
Mean	0
Standard deviation	1

Calculat

Note: The normal distribution table, found in the appendix of most statistics texts, is based on standard normal distribution, which has a mean of 0 and a standard deviation of 1. To produce outputs from a standard normal distribution with this calculator, set the mean equal to 0 and t standard deviation equal to 1.

Frequently-Asked Questions

Normal Distribution Calculator | Sample Problems

Instructions: To find the answer to a frequently-asked question, simply click on the question. I don't see the answer you need, try the Statistics Glossary or check out Stat Trek's tutorial on tł normal distribution.

- Why is the normal distribution so important?
- What is a standard normal distribution?
- What is a normal random variable?
- What is a standard score?
- What is a probability?

The point is that because $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2}$ is even ($f(-z) = f(z)$ because it is a function of z^2) the function $\Phi(z)$ also has (a more subtle) symmetry namely

$$\Phi(-a) = 1 - \Phi(a) \quad (*)$$

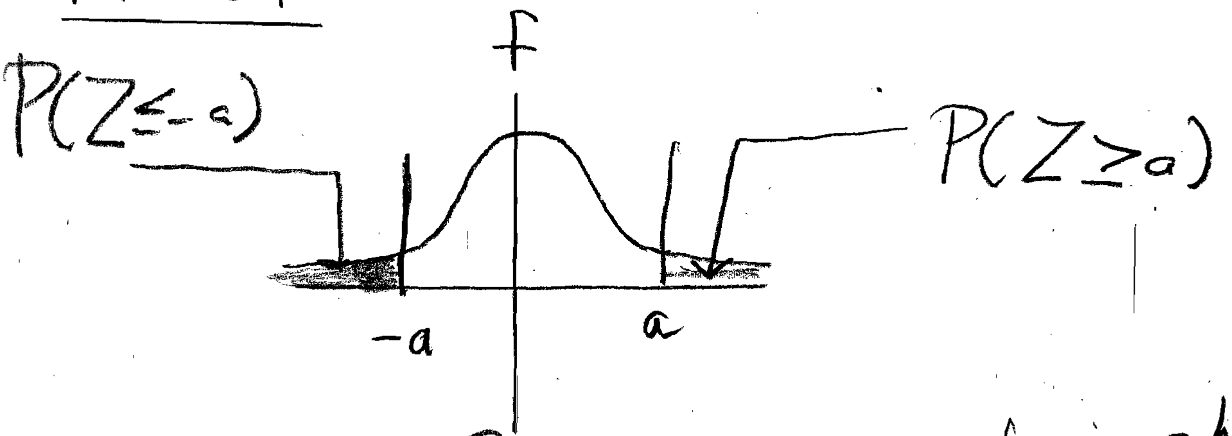
It is easiest to state and prove this in terms of probabilities

<u>Proposition</u>

$$P(Z \leq -a) = 1 - P(Z \leq a) \quad (**)$$

$(*)$ and $(**)$ are the same

Proof



Because $f(z)$ is symmetric about the y axis ($f(-z) = f(z)$) the two shaded areas have to be the same since one is the mirror image of other (where the y-axis is the mirror)

Hence

$$P(Z \le -a) = P(Z \ge a) = 1 - P(Z < a)$$

(because $(Z \ge a)$ and $(Z < a)$ are complements of each other).

But Z is continuous so

$$P(Z < a) = P(Z \le a) \text{ and}$$

$$P(Z \le -a) = 1 - P(Z \le a)$$

$\square$

Now we can do (b) given the answer to (a)

$$P(Z \le -1.25) = 1 - P(Z \le 1.25)$$
$$= 1 - .8944$$
$$= .1056$$

Now what about (c). We have

<u>Proposition</u> (The "<u>Handy Formula</u>")

$$P(-a \le Z \le a) = 2\Phi(a) - 1$$

<u>Proof</u>

$$P(-a \le Z \le a) = P(Z \le a) - P(Z < a)$$
$$= P(Z \le a) - P(Z \le -a)$$

we just proved this

$$= P(Z \le a) - (1 - P(Z \le a))$$
$$= 2P(Z \le a) - 1 = 2\Phi(a) - 1 \quad \square$$

So now we can do (c) using (e)

$$P(-1.25 \le Z \le 1.25) = 2\Phi(1.25) - 1$$

$$= 2(.8944) - 1 = .7888$$

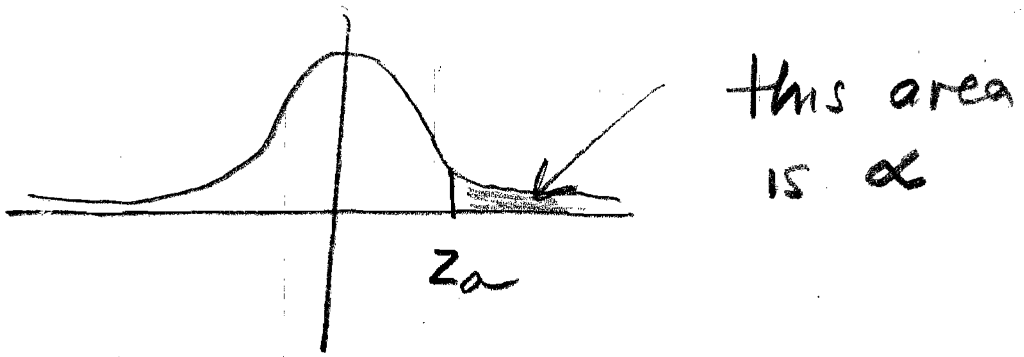
So we repeat - all we needed to do all three parts was the one value

$$\Phi(1.25) = P(Z \le 1.25) = .8944$$

The α-th critical value z_α of the standard normal

Let α be a real number between 0 and 1. We review the definition of the α-th critical value z_α (we have to change X to Z) from Lecture 11, pages 5, 6, 7

z_α is the number so that the vertical line $z = z_\alpha$ cuts off area α to the <u>right</u> under the graph of f of $f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.



Equivalently

$$P(Z \geq z_\alpha) = \alpha$$

or

$$1 - P(Z \leq z_\alpha) = \alpha$$

$$P(Z \leq z_\alpha) = 1 - \alpha$$

$$\Phi(z_\alpha) = 1 - \alpha$$

$$z_\alpha = \Phi^{-1}(1-\alpha)$$

Th

The values of z_α may be obtained from page 148 of the text or better, the back flap of the text, Table A-5.

α

ν	.1	.05	.025	.01	.005	.001	.0005
1							
2							
⋮			$t_{\alpha,\nu}$				
∞	1.282	1.645	- - - -				3.291

It may not look like it but the bottom row gives the values of z_α for $\alpha = .1, .05, .025, .01, .005, .001, .0005$

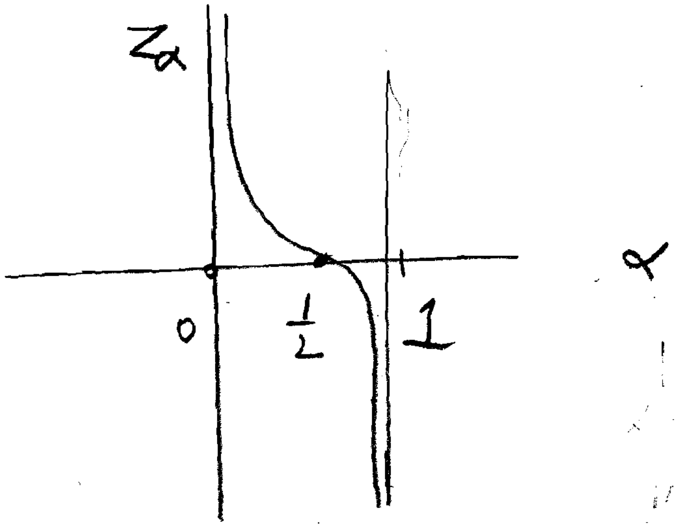
This is because

$$\lim_{\nu \to \infty} t_{\alpha,\nu} = z_\alpha$$

It will be important if you go further in statistics to think of z_α as a function of α, $z_\alpha = f(\alpha)$.

What is the graph of f?

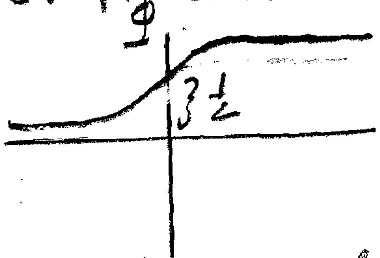
Here is the answer



<u>Hard Problem</u>

Prove this using operations on graphs and the formula $z_\alpha = \Phi^{-1}(1-\alpha)$

1. Start with the graph of $\Phi(z)$



2. Draw the graph of $\Phi^{-1}(z)$ then of $\Phi^{-1}(1-z)$ (you do this by "flipping" graphs).

Standardizing

Everybody has to learn how to do this!

When $X \sim N(\mu, \sigma^2)$ the probabilities $P(a \leq X \leq b)$ are computed by "standardizing" X. The procedure is based on

Proposition

If $X \sim N(\mu, \sigma^2)$ then the new random variable $Z = \dfrac{X-\mu}{\sigma}$

satisfies $Z \sim N(0,1)$

Remark 2 This may be too hard.

This is a linear change of continuous random variable. We haven't defined change of continuous random variable but we will say something now.

Here is the idea.

Write the density of X

as $$f(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

You have to put in the dx here.

Now substitute $z = \frac{x-\mu}{\sigma}$ or $x = \sigma z + \mu$

so $dx = \sigma\,dz$ so when we re-express the right-hand side in terms of z we get

$$\frac{1}{\sqrt{2\pi}\,\cancel{\sigma}}\, e^{-\frac{1}{2}z^2}\,\cancel{\sigma}\,dz = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{1}{2}z^2}\,dz$$ standard normal

In general when you make a change of variable from X to $Y = h(X)$ you take the density $f(x)\,dx$ of X and re-express everything in terms of y using $x = h^{-1}(y)$ so $dx = d(h^{-1}(y))$.

This is the idea but needs tightening up.

Now back to Stat 400 and what you absolutely have to know

Example

Suppose $X \sim N(40, (1.5)^2)$

Compute $P(39 \leq X \leq 42)$

Solution

Be careful: $\sigma^2 = (1.5)^2$ so $\sigma = 1.5$, you have to divide by $\sigma = 1.5$ below.

The desired probability is

$$P(39 \le X \le 42)$$

We subtract the mean $\mu = 40$ from EVERYTHING and divide EVERYTHING by $\sigma = 1.5$. This way we have an equality

$$P(39 \le X \le 42) = P\left(\frac{39-40}{1.5} \le \frac{X-40}{1.5} \le \frac{42-40}{1.5}\right)$$

these inequalities have the same set of solutions

because we have done the same thing to all terms in the inequalities on the left.

We obtain

$$P(39 \le X \le 42) = P\left(-\frac{1}{1.5} \le Z \le \frac{2}{1.5}\right)$$

$$= P\left(-\frac{2}{3} \le Z \le \frac{4}{3}\right)$$

$$= P(-.67 \le Z \le 1.33)$$

$$= \Phi(1.33) - \Phi(-.67)$$

from front flap

$$= .9082 - .2514$$

$$= .6568$$

Make sure you understand this computation completely.

In real-life problems you might not have a table available. Still you can give a good approximation to normal probabilities using the

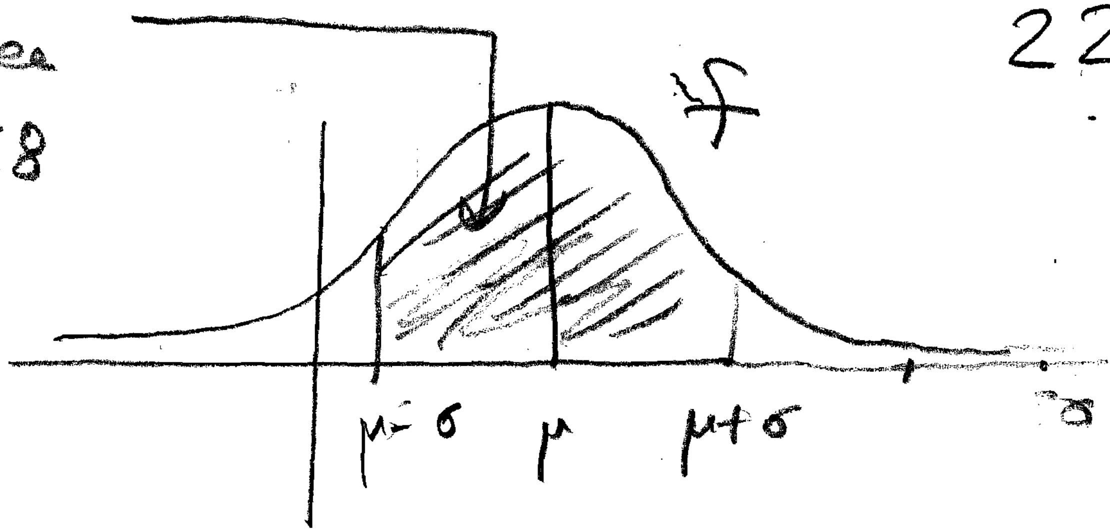
Two-Sided Rule of Thumb, page 151

Let $X \sim N(\mu, \sigma^2)$. We will give approximations for X to be within 1, 2 and 3 standard deviations of its mean

1) One standard deviation

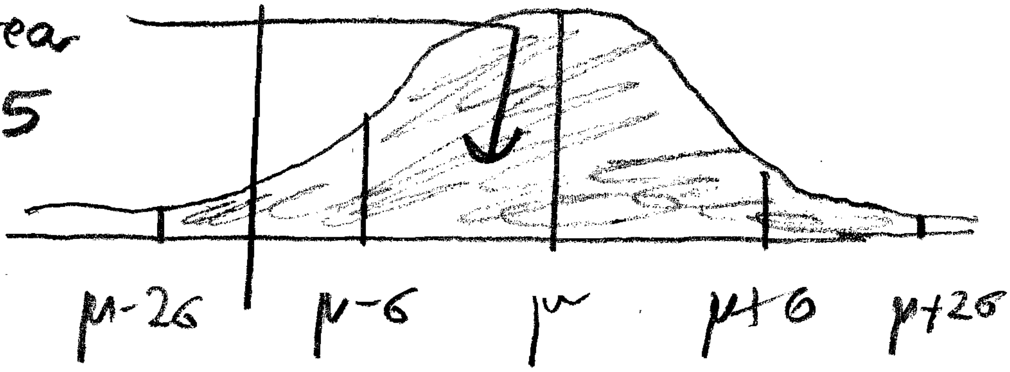
$$P(|X-\mu| \le \sigma) \approx .68$$

This area is $\approx .68$



(2) Two standard deviations

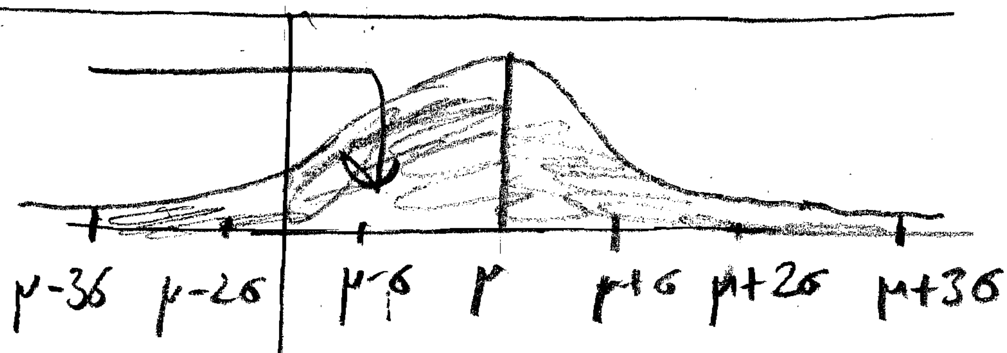
This area is $\approx .95$



$$P(\,|X-\mu| \le 2\sigma) \approx .95$$

(3) Three standard deviations

This area is $\approx .997$



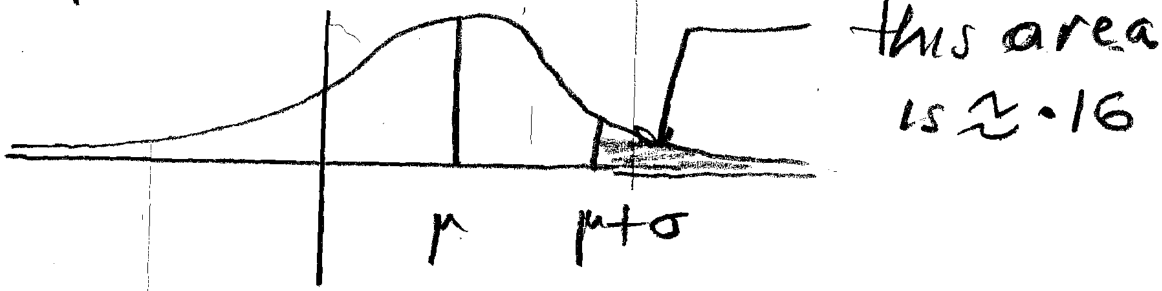
$$P(|X-\mu| \le 3\sigma) \approx .997$$

Consequence: (we will need these)

The One-Sided Rule of Thumb

One standard deviation

$$P(X-\mu > \sigma) \approx .16$$



Why

why

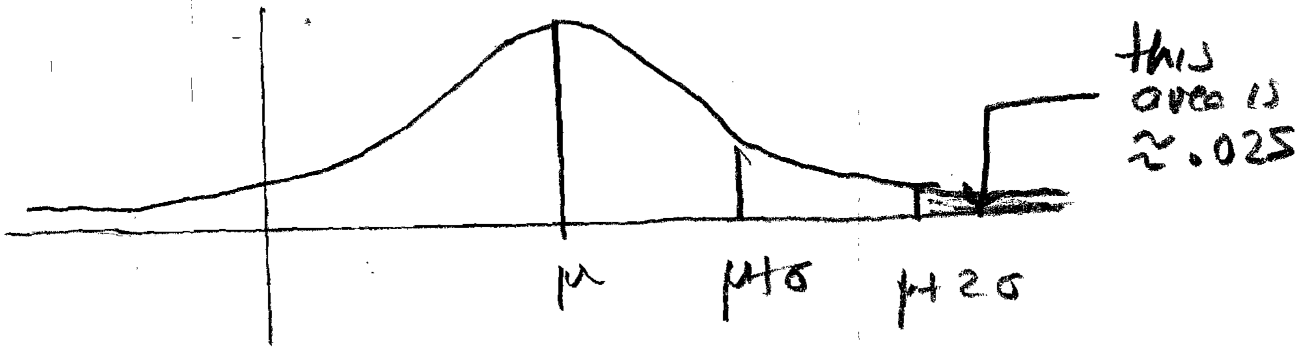
$$P(X-\mu > \sigma) = \frac{1}{2} P(|X-\mu| > \sigma).$$

$$= \frac{1}{2}\left(1 - P(|X-\mu| \le \sigma)\right)$$

$$\approx \frac{1}{2}(1 - .68) = \frac{1}{2}(.32) = .16$$

Two standard deviations

$$P(X-\mu > 2\sigma) \approx .025$$



Left to you.

The Normal Approximation to the Binomial

Recall

$X \sim Bin(n,p) \Longrightarrow E(X) = np$

and $V(X) = npq$

Theorem (Normal approximation to the binomial)

If $X \sim Bin(n,p)$ and n is large (relative to p and q) specifically $np \geq 10$ and $nq \geq 10$

then X is approximately normal.

Precisely if Y is the normal random variable with the same mean and variance as X

so $Y \sim (np, npq)$ then

for all a, b

$$P(a \leq X \leq b) \approx P(a \leq Y \leq b)$$

Problem

Professional mathematicians would not accept this theorem. There is no estimate on the error of the approximation.

On the cosmic scale $1 \approx 2$ but not in real life so "approximate" is too vague.

But in fact the above approximation is good to a large number of decimal places and there is a (complicated) estimate for the error.

Refinement (not that important)

"Correction for Continuity"

$$P(a \le X \le b) \approx P(a \le Y \le b) \quad (b)$$

But X is discrete and Y is continuous so X could assign a non-zero probability to the end a and to the other end b whereas X would assign zero probability to each end

For example

$$\underbrace{P(1 \le X \le 1)}_{P(X=1) = \binom{n}{1} p q^{n-1}} \approx \underbrace{P(1 \le Y \le 1)}_{P(Y=1)=0} \; ?$$

$\neq 0$

NO

So the right-hand side is <u>too small</u> so we make it bigger by pushing a $\frac{1}{2}$ unit to the left and pushing b $\frac{1}{2}$ unit to the right

See the text, pg 153, -here

$$P(\underbrace{10 \le X \le 10}_{X=10}) \approx P(9.5 \le Y \le 10.5)$$

<u>Bottom Line</u>

$$P(a \le X \le b) \approx P(a - \tfrac{1}{2} \le Y \le b + \tfrac{1}{2}) \quad \text{(bb)}$$

is better than (b)

The GRE Problem

(from Tim Darling, fall 1999)

The following problem was on the Graduate Records Exam in mathematics. As you will see it was pretty hard

Suppose a fair die is tossed 360 times. The probability a 6 comes up 70 or more times is

(A) $> .5$

(B) Between $(.16, .5)$

(C) Between $(.02, .16)$

(D) Between $(.01, .02)$

(E) $< .01$

Solution

Let X = # of sixes in 360 tosses

So success = 6 appears
failure = non 6 appears

so $X \sim Bin(360, \frac{1}{6})$

and ~~the~~ exact answer is

$$P(X \geq 70) \text{ with } X \sim Bin(360, \tfrac{1}{6})$$

But we need to choose between (A), (B), (C), (D) and (E)

So we need a number — if you had a laptop when you took the exam you wouldn't need what comes next.

So we use the <u>normal approximation</u> to X. We have $E(X) = np = (360)(\frac{1}{6}) = 60$

$$V(X) = npq = (np)q = (60)(\frac{5}{6}) = 50$$

So let $Y \sim N(60, 50)$

(Warning $\sigma = \sqrt{50} \approx 7.7$)

they should have planned better

So

$$P(X \geq 70) \approx P(Y \geq 70) \quad (*)$$

(we don't use the correction for continuity)

But we aren't done yet unless you have a table of normal probabilities with you

So we need the <u>one-sided rule of thumb</u>.

We have to compare the right-hand side of (*) to

$P(Y-\mu \geq \sigma) \approx .16$ and

$P(Y-\mu \geq 2\sigma) \approx .025$

Now $\mu = 60$ so

$P(Y \geq 70) = P(Y-60 \geq 10)$

So we got lucky, <u>10 is between $\sigma = 7.7$ and $2\sigma = 15.4$</u>

what we want

so

$$P(Y-\underset{60}{\mu} > \underset{7.7}{\sigma}) > P(Y-60 \geq 10) > P(Y-\underset{60}{\mu} > \underset{15.4}{2\sigma})$$

so

$$.16 > P(Y-60 > 10) > .025$$

<u>So the answer is (C)</u>