Lecture 30

Confidence Intervals for σ^2

Today we will discuss the material in Section 7.4.

Let $X_1, X_2, \dots, X_n$ be a random sample from a normal population with mean μ and variance σ^2.

$$\boxed{X \sim N(\mu, \sigma^2)} \dashrightarrow X_1, X_2, \dots, X_n$$

In this lecture we want to construct a $100(1-\alpha)\%$ confidence for σ^2.

We recall that S^2 is a point estimator for σ^2.

What is new here is that we are going to make a "multiplicative confidence interval".

Here is the idea. We want a random interval that has the point estimator S^2 in the interior

Now given a number x there are two ways to make an interval $I(x)$ that has x in its interior.

1. The additive method

Choose two positive numbers c_1 and c_2.

Put $I(x) = (x - c_1, x + c_2)$.

2. The multiplicative method

Choose a number $c_1 < 1$ and another number $c_2 > 1$. Put

$$I(x) = (c_1 x, c_2 x).$$

We will use the second method now. The clue to why we do this is that $S^2 > 0$.

First we need to know the probability distribution of the point estimator S^2. We have already seen this

<u>Theorem A</u> (pg 278)

$$V = \left(\frac{n-1}{\sigma^2}\right) S^2 \sim \chi^2(n-1) \qquad (*)$$

Now we can give the confidence interval.

<u>Theorem B</u>

The random interval $\left(\frac{n-1}{\chi^2_{\alpha/2, n-1}} S^2, \frac{n-1}{\chi^2_{1-\alpha/2, n-1}} S^2 \right)$

is a $100(1-\alpha)\%$ confidence random interval for the population variance σ^2 from a normal population

Remark

It must be true (see page 2) that

$$c_1 = \frac{n-1}{\chi^2_{\alpha/2, n-1}} < 1 \quad \text{and}$$

$$c_2 = \frac{n-1}{\chi^2_{1-\alpha/2, n-1}} > 1.$$

I have never checked this.

Now we prove Theorem B. We must prove

$$P\left(\sigma^2 \in \left(\frac{n-1}{\chi^2_{\alpha/2, n-1}} S^2, \frac{n-1}{\chi^2_{1-\alpha/2, n-1}} S^2\right)\right) = 1$$

$$LHS = P\left(\frac{n-1}{\chi^2_{\alpha/2, n-1}} S^2 < \sigma^2, \sigma < \frac{n-1}{\chi^2_{1-\alpha/2, n-1}} S^2\right)$$

Now we manipulate the two resulting inequalities to get V so we can use $\circledast$)

$$P\left(\frac{n-1}{\chi^2_{\alpha/2,n-1}} S^2 < \sigma^2,\ \sigma^2 < \frac{n-1}{\chi^2_{1-\alpha/2,n-1}} S^2\right)$$

swap and make V

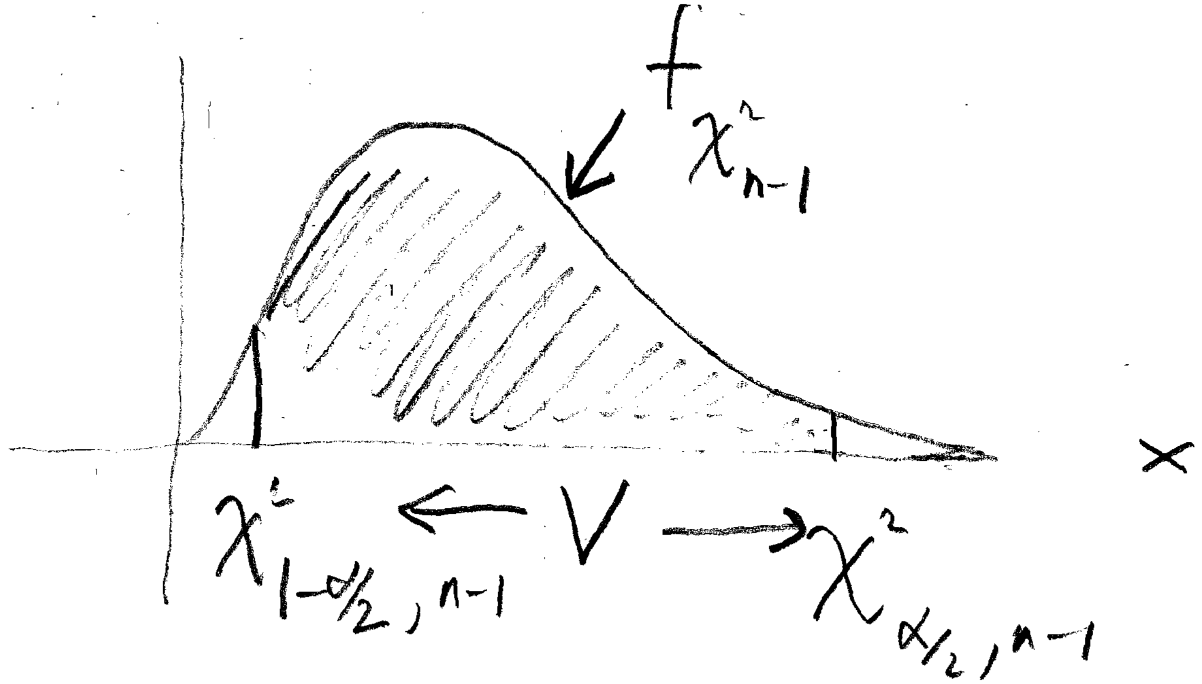
$$= P\left(\frac{n-1}{\sigma^2} S^2 < \chi^2_{\alpha/2,n-1},\ \chi^2_{1-\alpha/2,n-1} < \frac{n-1}{\sigma^2} S^2\right)$$

$$= P\left(V < \chi^2_{\alpha/2,n-1},\ \chi^2_{1-\alpha/2,n-1} < V\right)$$

$$= P\left(\chi^2_{1-\alpha/2,n-1} < V < \chi^2_{\alpha/2,n-1}\right)$$

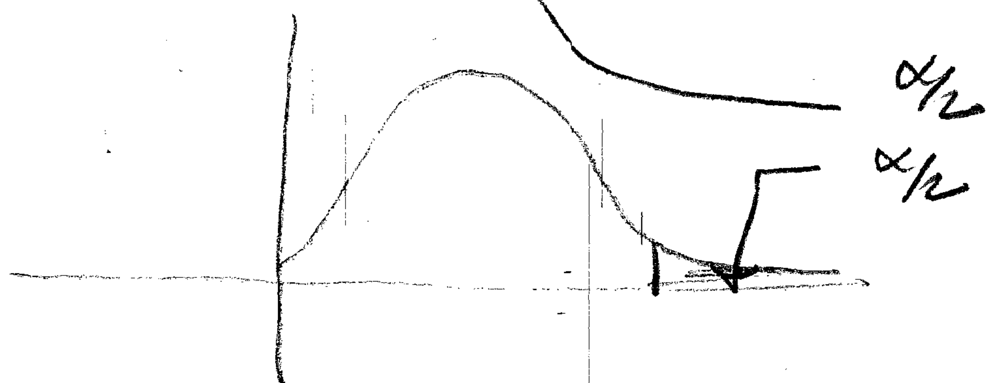
MAKE A PICTURE

= the shaded area

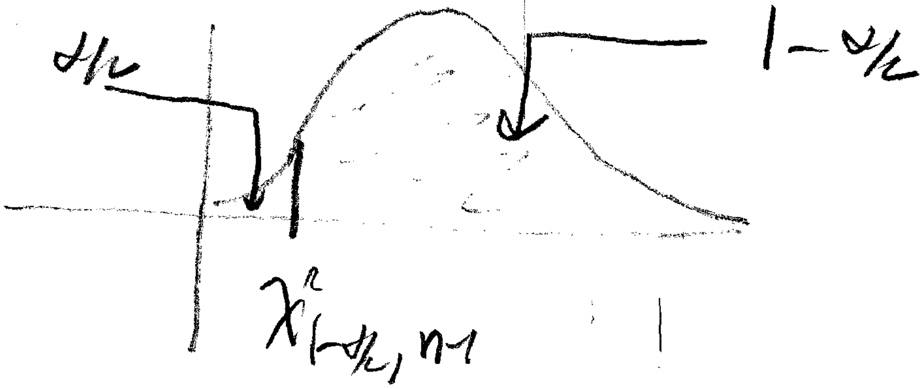


$= 1 - ($  $+$  $)$

Now



and



$= 1 - (\alpha/2 + \alpha/2) = 1 - \alpha \quad \square$

<u>Questun</u>

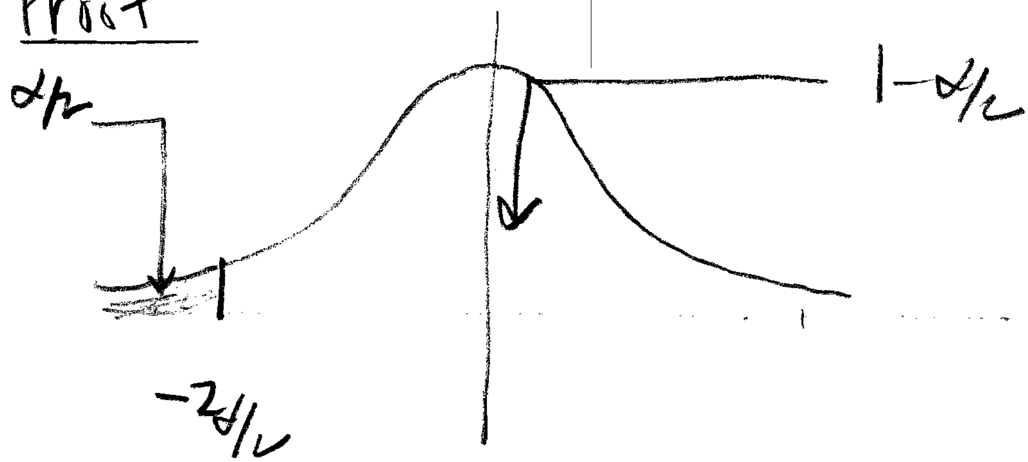
Why do we need the strange $\chi^2_{1-\alpha/2,n-1}$? This is because the χ^2 density curve does not have the symmetry that the z-density and t-densities did. In all three cases we need something that cut off $\alpha/2$ on the <u>left</u> under the density curve so $1-\alpha/2$ on the <u>right</u>. For the z-curve $-z_{\alpha/2}$ did the job

In other words

Lemma

$$z_{1-\alpha/2} = -z_{\alpha/2}$$

Proof



so $-z_{\alpha/2}$ cuts off $1-\alpha/2$ to the right so $-z_{\alpha/2} = z_{1-\alpha/2}$

$\square$

The Upper-Tailed $100(1-\alpha)\%$ Confidence Interval for σ^2

Theorem

$\left(\frac{n-1}{\chi^2_{\alpha, n-1}} S^2, \infty\right)$ is a $100(1-\alpha)\%$ confidence interval for σ^2

Proof It could be on the final – do it yourself.

Remark

As usual we took the lower limit from the two-sided interval and changed $\alpha/2$ to α.

The Lower-Tailed $100(1-\alpha)\%$ Confidence Interval for σ^2.

Since S^2 is always positive

$P(S^2 \in (-\infty, 0)) = 0$ so

the negative axis will not appear.

Lower tailed multiplicative intervals go down to 0 not $-\infty$. Another (philosophical) way to look at it is

$$\underbrace{\begin{array}{c}\text{additive group of } \mathbb{R} \\ (-\infty, \infty)\end{array}}_{\text{additive world}} \xrightarrow{e^x} \underbrace{\begin{array}{c}\text{multiplicative group of positive numbers} \\ (0, \infty)\end{array}}_{\text{multiplicative world}}$$

We are in the multiplicative world

Theorem

$$\left(0, \frac{n-1}{\chi^2_{1-\alpha, n-1}} S^2\right)$$ is a

$100(1-\alpha)\%$ confidence interval for σ^2.

Proof Do it yourself.

Remark

$$\left(-\infty, \frac{n-1}{\chi^2_{1-\alpha, n-1}} S^2\right)$$ is also a

$100(1-\alpha)\%$ confidence interval for σ^2

but the $(-\infty, 0)$ is "wasted space". Remember, small intervals are better.

Confidence Intervals for the Standard Deviation

Note that if $a > 0$, $b > 0$ and $x > 0$ then

$$a \le x \le b \iff \sqrt{a} \le \sqrt{x} \le \sqrt{b}.$$

So

$$\frac{n-1}{\chi^2_{\alpha/2,\, n-1}} S^2 \le \sigma^2 \le \frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}} S^2$$

$$\iff \sqrt{\frac{n-1}{\chi^2_{\alpha/2,\, n-1}}}\, S \le \sigma \le \sqrt{\frac{n-1}{\chi^2_{(1-\alpha/2),\, n-1}}}\, S$$

Hence

$$P\left(\sqrt{\frac{n-1}{\chi^2_{\alpha/2,\, n-1}}}\, S < \sigma < \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}}}\, S\right) = P\left(\frac{n-1}{\chi^2_{\alpha/2,\, n-1}} S^2 \le \sigma^2 \le \frac{n-1}{\chi^2_{1-\alpha/2,\, n-1}} S^2\right)$$

from pg3

$$= 1-\alpha$$

In other words

$$P\left(\sigma \in \left(\sqrt{\frac{n-1}{\chi^2_{\alpha/2,n-1}}}\,S,\ \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,n-1}}}\,S\right)\right) = 1-\alpha$$

and we have

Theorem

The random interval

$$\left(\sqrt{\frac{n-1}{\chi^2_{\alpha/2,n-1}}}\,S,\ \sqrt{\frac{n-1}{\chi^2_{1-\alpha/2,n-1}}}\,S\right)$$

is a $100(1-\alpha)\%$ confidence interval for the standard deviation σ in a normal population.

Problem

Write down the upper and lower-tailed confidence intervals for σ.

(hint: just take the square roots of the endpoints of those for σ^2)