The confidence interval formulas for the mean in an normal distribution when σ is known

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1 Introduction

In this lecture we will derive the formulas for the symmetric two-sided confidence interval and the lower-tailed confidence intervals for the mean in a normal distribution *when the variance σ^2 is known*. At the end of the lecture I assign the problem of proving the formula for the upper-tailed confidence interval as HW 12. We will need the following theorem from probability theory that gives the distribution of the statistic $\overline{X}$ - the point estimator for μ.

Suppose that $X_1, X_2, \ldots, X_n$ is a random sample from a normal distribution with mean μ and variance σ^2. We assume μ is unknown but σ^2 is known. We will need the following theorem from Probability Theory.

Theorem 1. *$\overline{X}$ has normal distribution with mean μ and variance σ^2/n.*
Hence the random variable $Z = (\overline{X} - \mu)/\frac{\sigma}{\sqrt{n}}$ has standard *normal distribution.*

2 The two-sided confidence interval formula

Now we can prove the theorem from statistics giving the required confidence interval for μ. Note that it is symmetric around $\overline{X}$. There are also asymmetric two-sided confidence intervals. We will discuss them later. This is one of the basic theorems that you have to learn how to prove.

Theorem 2. *The random interval $(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$ is a $100(1-\alpha)\%$-confidence interval for μ.*

Proof. We are required to prove

$$P(\mu \in (\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})) = 1 - \alpha.$$

We have

$$\begin{aligned}
LHS &= P(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu, \mu < \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) = P(\overline{X} - \mu < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ -z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \overline{X} - \mu) \\
&= P(\overline{X} - \mu < z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} - \mu > -z_{\alpha/2}\frac{\sigma}{\sqrt{n}}) \\
&= P((\overline{X} - \mu)/\frac{\sigma}{\sqrt{n}} < z_{\alpha/2}\ ,\ (\overline{X} - \mu)/\frac{\sigma}{\sqrt{n}} > -z_{\alpha/2}) \\
&= P(Z < z_{\alpha/2}\ ,\ Z > -z_{\alpha/2}) = P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ and the observed value $(\overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}, \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$ for the confidence (random) interval $(\overline{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\ ,\ \overline{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$. The observed value of the confidence (random) interval is also called the two-sided $100(1-\alpha)\%$ confidence interval for μ.

3 The lower-tailed confidence interval

In this section we will give the formula for the lower-tailed confidence interval for μ.

Theorem 3. *The random interval* $(-\infty\ ,\ \overline{X} + z_\alpha\frac{\sigma}{\sqrt{n}})$ *is a* $100(1-\alpha)\%$*-confidence interval for* μ.

Proof. We are required to prove

$$P(\mu \in (-\infty\ ,\ \overline{X} + z_\alpha\frac{\sigma}{\sqrt{n}})) = 1 - \alpha.$$

We have

$$\begin{aligned}
LHS &= P(\mu < \overline{X} + z_\alpha\frac{\sigma}{\sqrt{n}}) = P(-z_\alpha\frac{\sigma}{\sqrt{n}} < \overline{X} - \mu) \\
&= P(-z_\alpha < (\overline{X} - \mu)/\frac{\sigma}{\sqrt{n}}) \\
&= P(-z_\alpha < Z) \\
&= 1 - \alpha
\end{aligned}$$

To prove the last equality draw a picture - I want *you* to draw the picture on tests and the homework. □

Once we have an actual sample $x_1, x_2, \ldots, x_n$ we obtain the observed value $\overline{x}$ for the random variable $\overline{X}$ and the observed value $(-\infty, \overline{x} + z_\alpha \frac{\sigma}{\sqrt{n}})$ for the confidence (random) interval $(-\infty, \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}})$. The observed value of the confidence (random) interval is also called the lower-tailed $100(1-\alpha)\%$ confidence interval for μ.

The number random variable $\overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}$ or its observed value $\overline{x} + z_\alpha \frac{\sigma}{\sqrt{n}}$ is often called a confidence *upper bound* for μ because

$$P(\mu < \overline{X} + z_\alpha \frac{\sigma}{\sqrt{n}}) = 1 - \alpha.$$

4 The upper-tailed confidence interval for μ

Homework 12 (to be handed in on Monday, Nov.28) is to prove the following theorem.

Theorem 4. *The random interval* $(\overline{X} - z_\alpha \frac{\sigma}{\sqrt{n}}\ ,\ \infty)$ *is a* $100(1-\alpha)\%$ *confidence interval for* μ.