Stat 400 MIDTERM 2 April 19, 2012

1. Suppose X and Y are random variables defined on the same sample space with the following joint probability mass function.

X \ Y	0	1	2
0	1/4	1/4	1/8
1	1/4	0	1/8

(a) Compute the probability mass functions of the random variables X and Y.
(b) Are X and Y independent?
(c) Compute the probability mass function of the random variable $W = X + Y$.
(d) Compute the covariance $Cov(X, Y)$.
(e) Compute the correlation $\rho_{X,Y}$.

(25 points)

2. Suppose a system of six components is arranged as follows.

Assume the lifetime of each component is exponentially distributed with parameter λ and that the components function independently. More precisely let $X_i, 1 \leq i \leq 6$, be the random variable defined by $(X_i = t)$ means the i-th component dies at time t. So $P(X_i > t) = e^{-\lambda t}$ and $P(X_i < t) = 1 - e^{-\lambda t}$. Find the cumulative density function for the system lifetime. More precisely , if X is the random variable defined by $(X = t)$ means the system dies at time t find $P(X \leq t)$.
(Hint: $(X_i \leq t)$ means the i-th component is already dead at time t and $(X_i > t)$ means the i-th component is still alive at time t.)

(10 points)

Turn the page

3. Suppose that X and Y are discrete random variables defined on the same sample space S. Suppose both X and Y have Bernoulli distribution with parameter p so

$$P(X = 0) = P(Y = 0) = q = 1 - p \text{ and } P(X = 1) = P(Y = 1) = p.$$

(a) Find the joint probability mass function $p_{X,Y}(x, y)$ of X and Y.
(Hint: make a 2 by 2 matrix with margins, use the formulas above to fill in the margins, then use independence to fill in the matrix.)
(b) Find the probability mass function of the random variable $W = X + Y$.
(c) W is one of the distributions you have seen many times in this course, which one?

(5 points, parts (a) and (b) are each worth 2 points, (c) is worth 1 point)

4. Suppose that X and Y are discrete random variables defined on the same sample space S. Suppose both X and Y both have uniform distribution on the interval $[0, 1]$ so

$$f_X(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases} \text{ and } f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the joint probability density function $f_{X,Y}(x, y)$ of X and Y.

(Hint: Since X and Y are independent $f_{X,Y}(x, y)$ is the product of $f_X(x)$ and $f_Y(y)$. This product is not so easy to compute, the point is to figure out where it is zero and where it is one in the plane. Zero and one are the only values taken by $f_{X,Y}(x, y)$.)

(b) Find $P(1/2 \leq X \leq 3/4, 1/2 \leq Y \leq 3/4)$.

(5 points, part(a) is worth 3 points and part (b) is worth 2 points)

5. Suppose X has the linear density so

$$f_X(x) = \begin{cases} 2x, & 0 \leq x \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find a change of variable $Y = h(X)$ such that Y has uniform distribution on $[0, 1]$, that is

$$f_Y(y) = \begin{cases} 1, & 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Verify using the "Engineer's Way" that the Y you come up with does in fact have uniform distribution.

(5 points)